

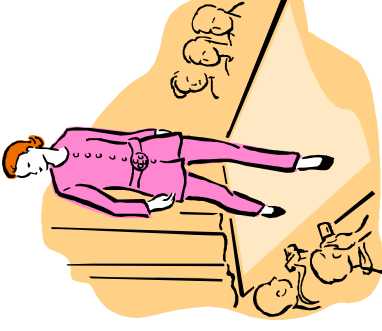
A Principled Approach to MILP Problem Formulation

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ICS 2009

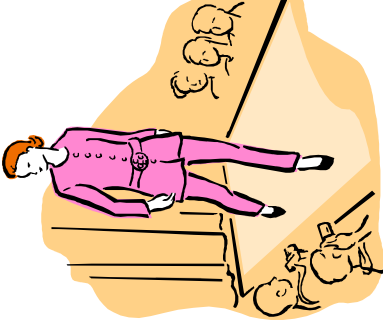
Proposal

- MILP modeling is an art, but it need not be unprincipled.



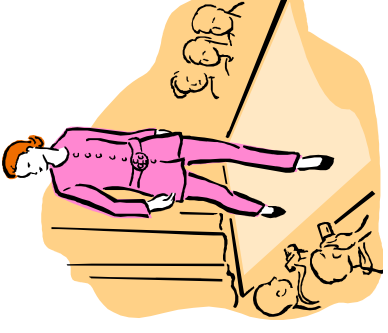
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- MILP modeling is an art, but it need not be unprincipled.
- It has two basic components:
 - Disjunctive modeling of subsets of continuous space.
 - Knapsack modeling of counting ideas.

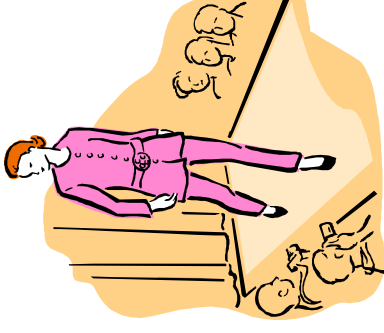


Proposal

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- It has two basic components:
 - Disjunctive modeling of subsets of continuous space.
 - Knapsack modeling of counting ideas.
- MILPs can model subsets of continuous space that are unions of polyhedra.
 - ...that is, represented by disjunctions of linear systems.



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- MILP modeling is an art, but it need not be unprincipled.
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 - Disjunctive modeling of subsets of continuous space.
 - Knapsack modeling of counting ideas.
- MILPs can model subsets of continuous space that are unions of polyhedra.
 - ...that is, represented by disjunctions of linear systems.
- So a principled approach is to analyze the problem as

disjunctions of linear systems + integer knapsack inequalities

Proposal

- **Jeroslow's Representability Theorem** provides theoretical basis for disjunctive modeling.
- “Bounded MILP representability” assumes bounded integer variables.
- This is inadequate for knapsack modeling.

Proposal

- **Jeroslow's Representability Theorem** provides theoretical basis for disjunctive modeling.
 - “Bounded MILP representability” assumes bounded integer variables.
 - This is inadequate for knapsack modeling.
- We will **generalize** Jeroslow's theorem.
 - Knapsack modeling accommodated.
 - Integer variables can be unbounded.

Outline

- **Bounded mixed integer representability**
 - Bounded representability theorem.
 - Convex hull formulation
 - *Example:* Fixed charge problem
 - *Example:* Facility location
 - *Example:* Lot sizing with setup costs
- **Big-M disjunctive formulation**
 - *Example:* Health care benefits

Outline

- **General mixed integer representability**
 - Knapsack models
 - General representability theorem.
 - Convex hull formulation
 - *Example: Facility location*
 - *Example: Freight packing and transfer*
 - Research issues

Bounded MILP Representability

Bounded representability theorem

Definition of R. Jeroslow:

A subset S of \mathbb{R}^n is **bounded MILP representable** if S is the projection onto x of the feasible set of some MILP constraint set of the form

$$Ax + Bz + Dy \geq b$$

$$x \in \mathbb{R}^n, z \in \mathbb{R}^m$$

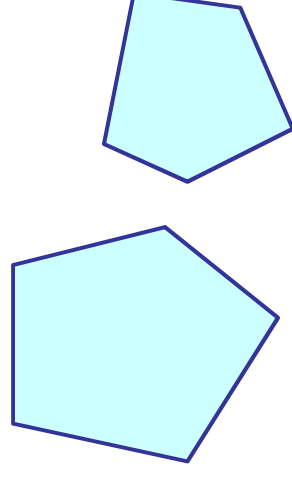
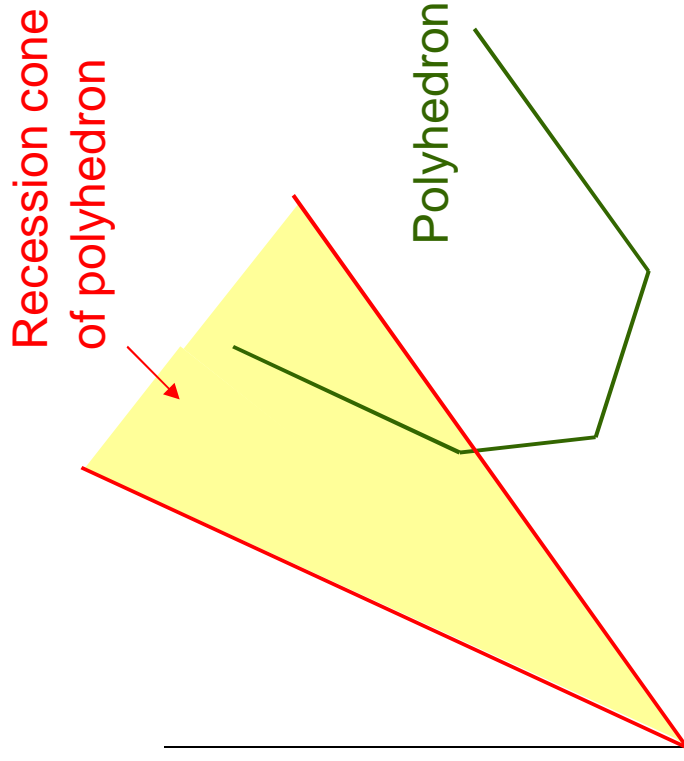
$$y \in \{0, 1\}^p$$

Auxiliary
continuous
variables can
be used

Bounded general
integer variables
can be encoded as
0-1 variables

Bounded representability theorem

Theorem (Jeroslow). A subset of continuous space is bounded MILP representable if and only if it is the union of finitely many polyhedra having the same recession cone.



Union of polyhedra with the same recession cone (in this case, the origin)

Convex hull formulation

Start with a disjunction of linear systems to represent the union of polyhedra.

The k th polyhedron is $\{x \mid A^k x \geq b\}$

Introduce a 0-1 variable y_k that is 1 when x is in polyhedron k .

Disaggregate x to create an x^k for each k .

$$\bigvee_k (A^k x \geq b^k)$$

$$A^k x^k \geq b^k y_k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k x^k$$

$$y_k \in \{0,1\}$$

Convex hull formulation

Start with a disjunction of linear systems to represent the union of polyhedra.

The k th polyhedron is $\{x \mid A^k x \geq b\}$

Introduce a 0-1 variable y_k that is 1 when x is in polyhedron k .

Disaggregate x to create an x^k for each k .

Every bounded MILP representable set has a model of this form.

$$\bigvee_k (A^k x \geq b^k)$$

$$A^k x^k \geq b^k y_k, \text{ all } k$$

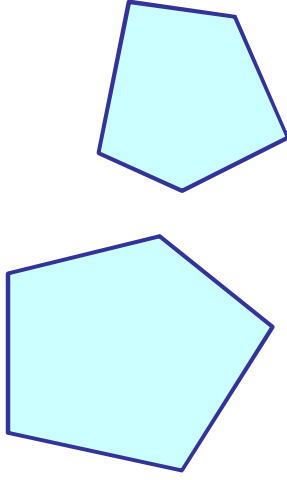
$$\sum_k y_k = 1$$

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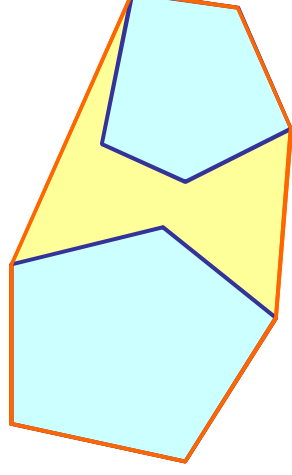
$$y_k \in \{0,1\}$$

Convex Hull Formulation

- The continuous relaxation of this disjunctive MILP provides a **convex hull relaxation** of the disjunction.
- Strictly, it describes the **closure** of the convex hull.



Union of polyhedra

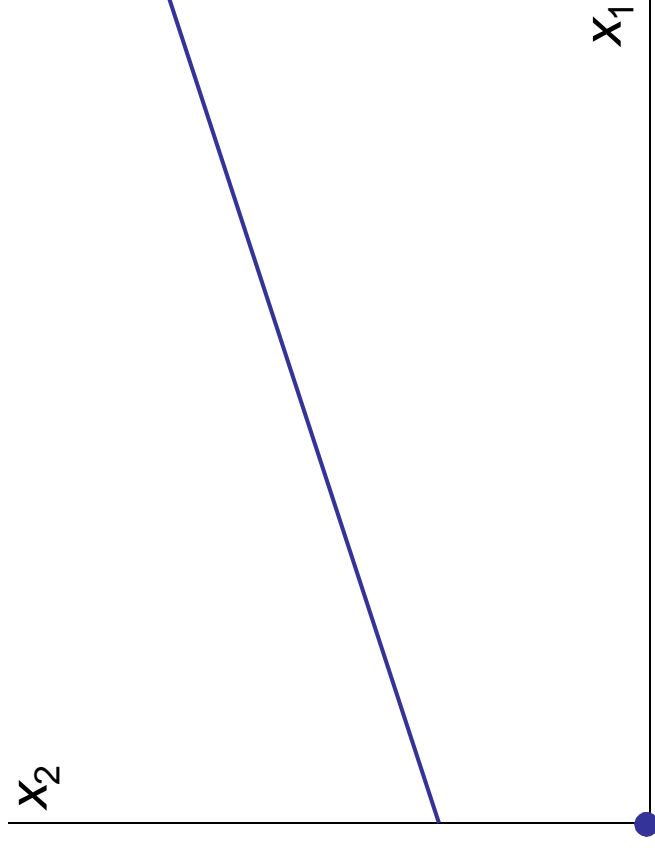


Convex hull relaxation
(tightest linear relaxation)

Example: Fixed charge function

$$\begin{aligned} & \min x_2 \\ & x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases} \\ & x_1 \geq 0 \end{aligned}$$

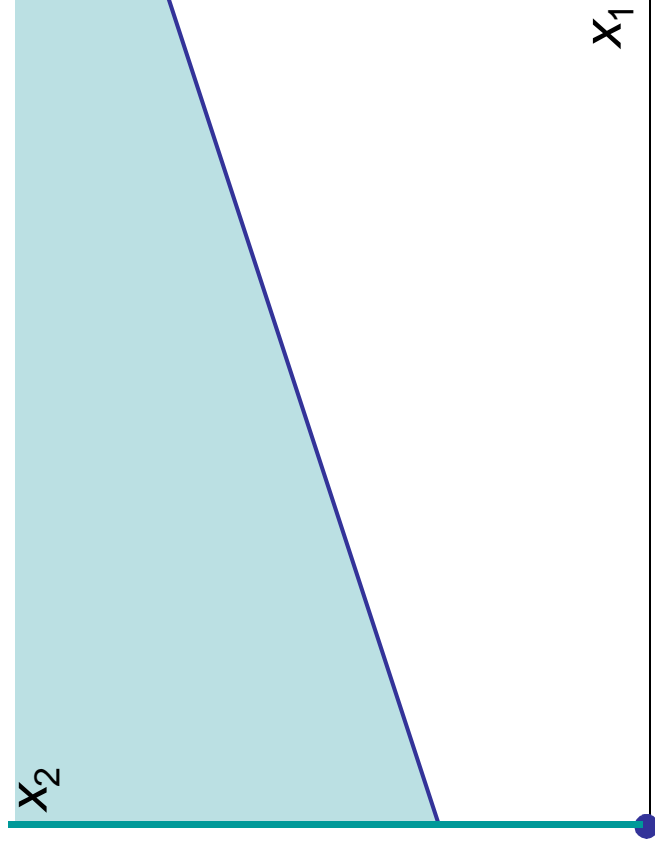
Minimize a fixed charge function:



Fixed charge problem

Minimize a fixed charge function:

$$\begin{aligned} \min \quad & x_2 \\ & x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases} \\ & x_1 \geq 0 \end{aligned}$$

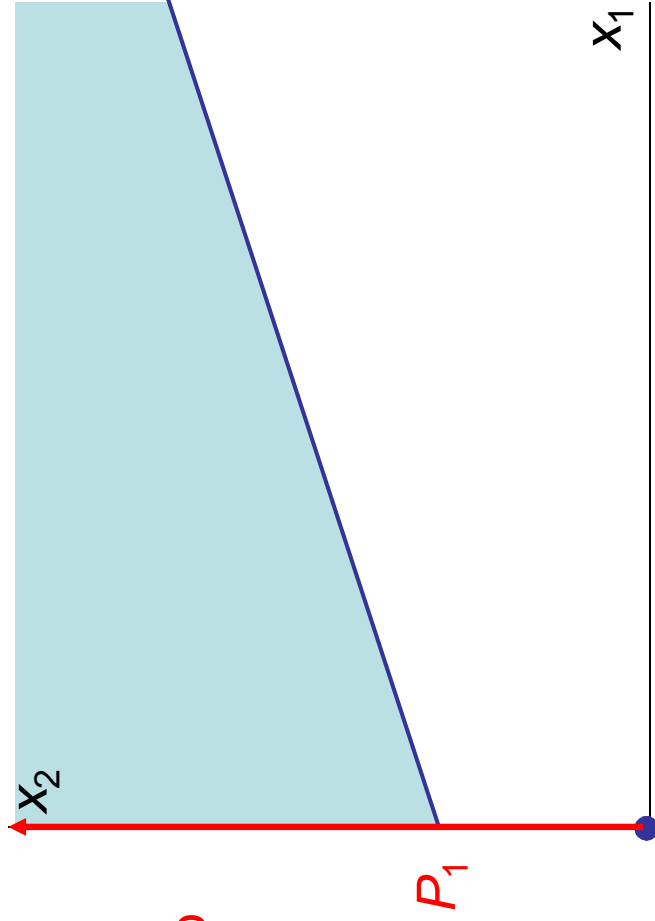


Feasible set
(epigraph)

Fixed charge problem

Minimize a fixed charge function:

$$\begin{aligned} \min \quad & x_2 \\ & x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases} \\ & x_1 \geq 0 \end{aligned}$$

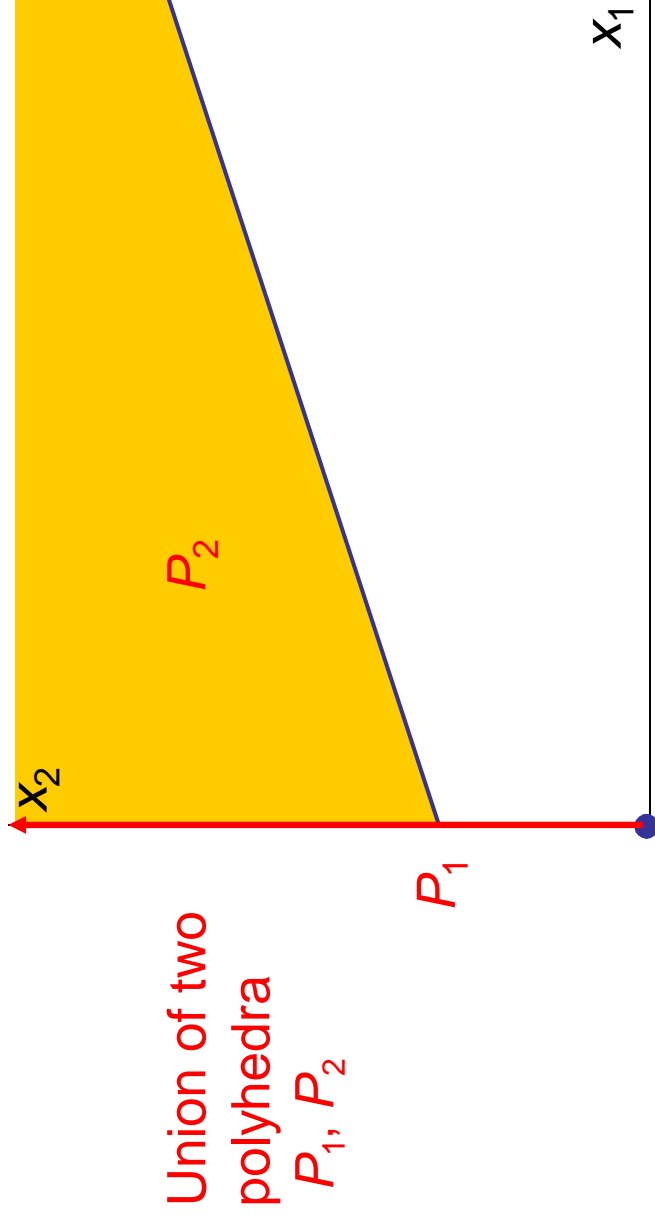


Union of two
polyhedra
 P_1, P_2

Fixed charge problem

Minimize a fixed charge function:

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases} \\ & x_1 \geq 0 \end{aligned}$$

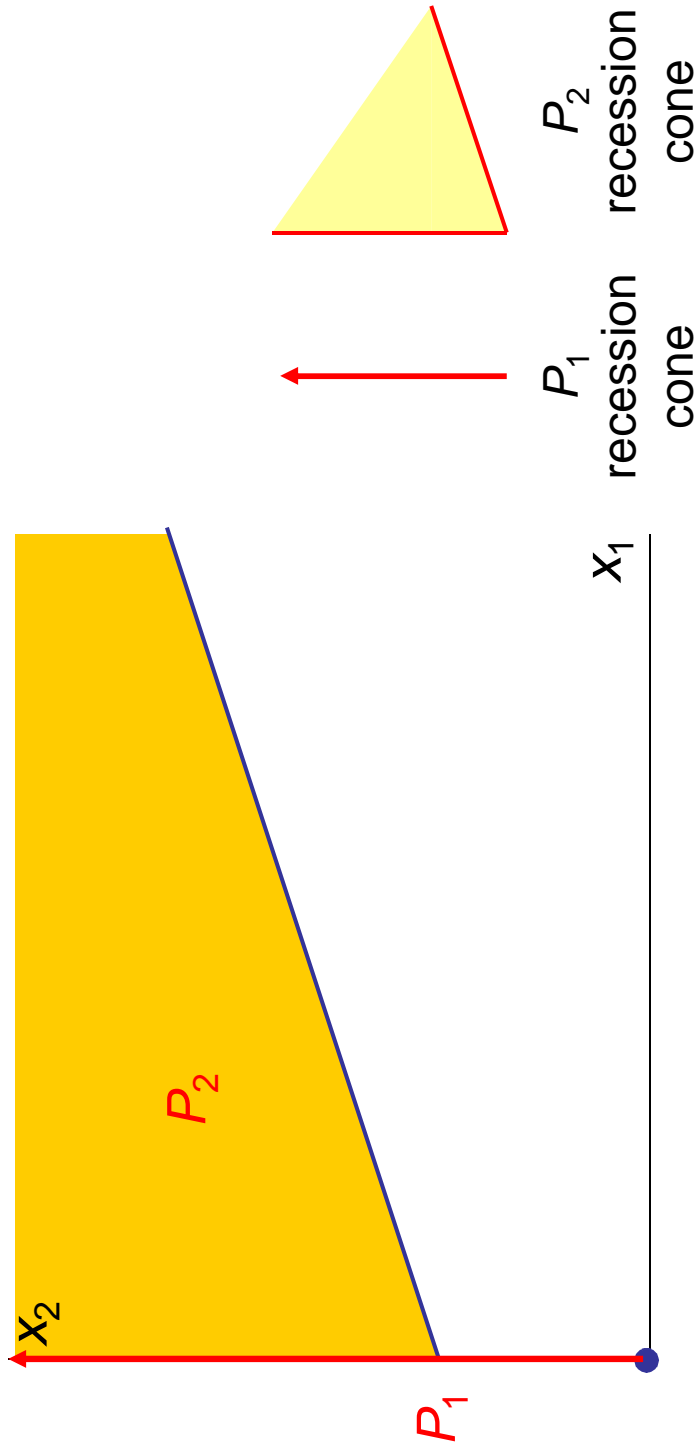


Fixed charge problem

Minimize a fixed charge function:

$$\begin{aligned} \min \quad & x_2 \\ x_2 \geq \quad & \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases} \\ x_1 \geq \quad & 0 \end{aligned}$$

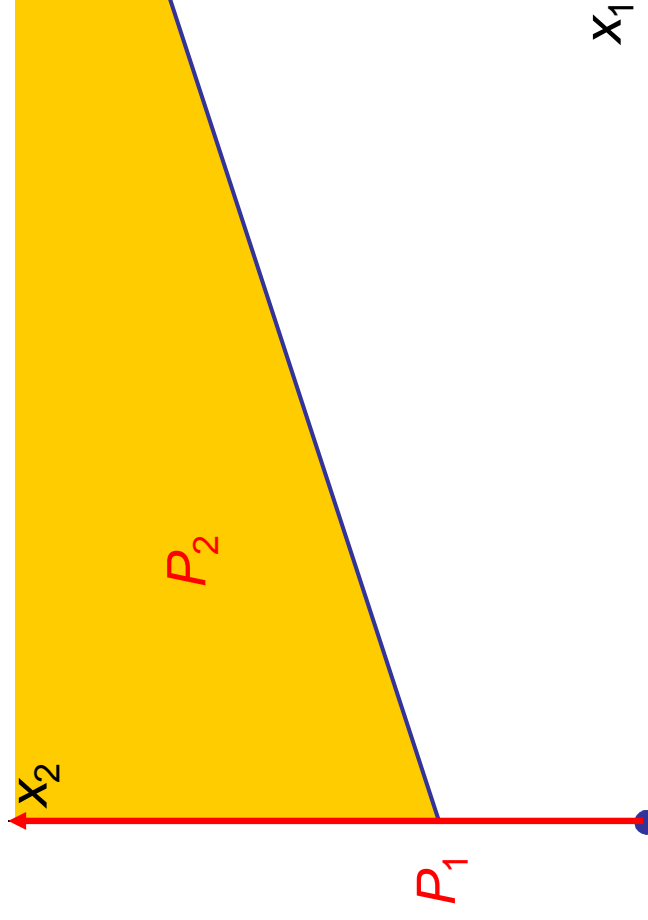
The polyhedra have different recession cones.



Fixed charge problem

Disjunctive model describes
convex hull relaxation but
not the feasible set.

$$\min x_2$$
$$\left(\begin{array}{l} x_1 = 0 \\ x_2 \geq 0 \end{array} \right) \vee \left(\begin{array}{l} x_1 \geq 0 \\ x_2 \geq f + cx_1 \end{array} \right)$$



Fixed charge problem

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min x_2 \quad \vee \left(\begin{array}{l} x_1 = 0 \\ x_2 \geq 0 \end{array} \right) \vee \left(\begin{array}{l} x_1 \geq 0 \\ x_2 \geq f + cx_1 \end{array} \right)$$

Introduce a 0-1 variable y_k that is 1 when x is in polyhedron k .

Disaggregate x to create an x^k for each k .

$$\begin{array}{ll} \min x_2 & \\ x_1^1 = 0 & x_1^2 \geq 0 \\ x_2^1 \geq 0 & -cx_1^2 + x_2^2 \geq fy_2 \\ y_1 + y_2 = 1, & y_k \in [0,1] \\ x_1 = x_1^1 + x_1^2, & x_2 = x_2^1 + x_2^2 \end{array}$$

Fixed charge problem

Start with a disjunction of linear systems to represent the union of polyhedra

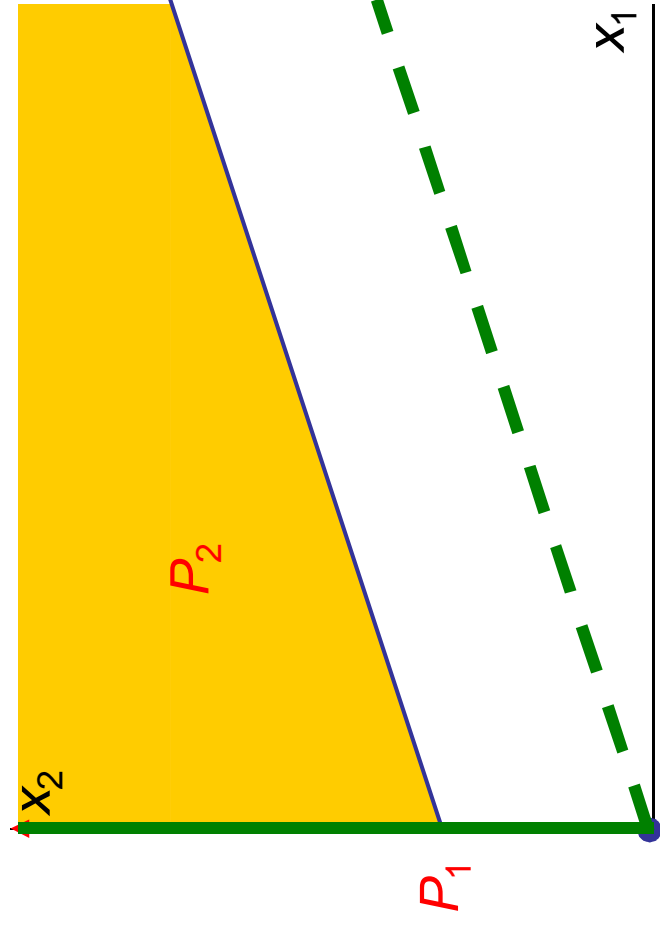
$$\min x_2 \quad \left(\begin{array}{l} x_1 = 0 \\ x_2 \geq 0 \end{array} \right) \vee \left(\begin{array}{l} x_1 \geq 0 \\ x_2 \geq f + cx_1 \end{array} \right)$$

Simplify the model.

$$\begin{array}{l} \min x_2 \\ x_1 \geq 0 \\ x_2 \geq cx_1 + fy \\ y \in [0,1] \end{array}$$

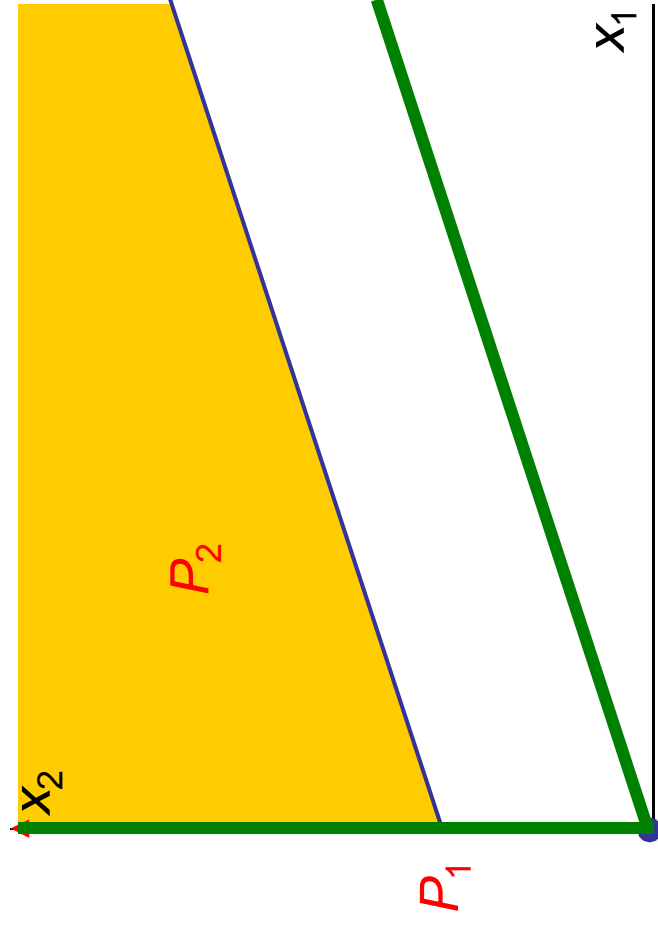
$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & x_1 \geq 0 \\ & x_2 \geq cx_1 + fy \\ & y \in [0,1] \end{aligned}$$

The convex hull is this.



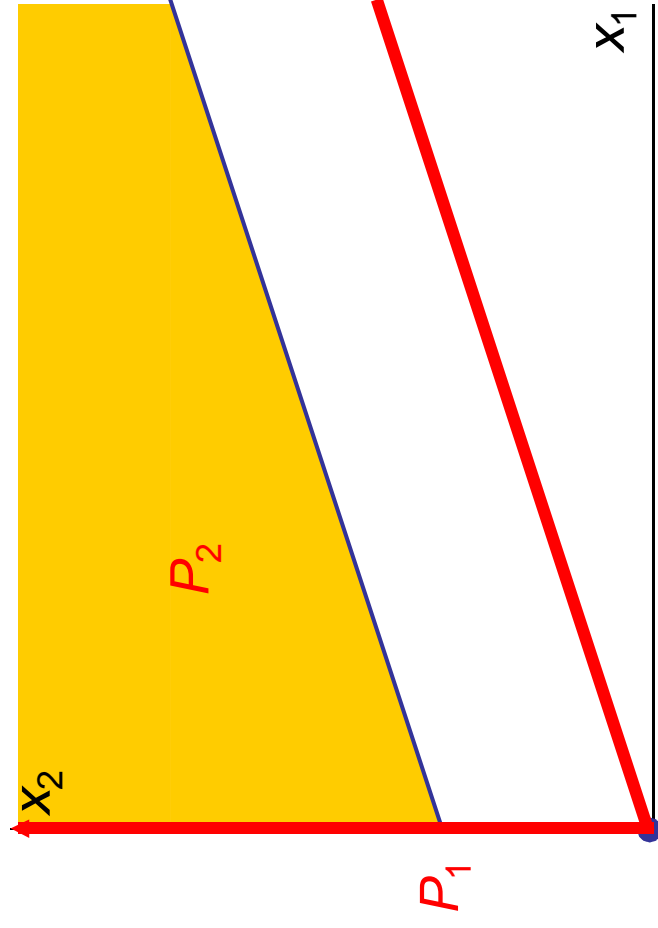
$$\begin{aligned} \min \quad & x_2 \\ & x_1 \geq 0 \\ & x_2 \geq cx_1 + fy \\ & y \in [0,1] \end{aligned}$$

Relaxation correctly describes
closure of convex hull



$$\begin{aligned} \min \quad & x_2 \\ & x_1 \geq 0 \\ & x_2 \geq cx_1 + fy \\ & y \in \{0,1\} \end{aligned}$$

But MILP model does not describe feasible set



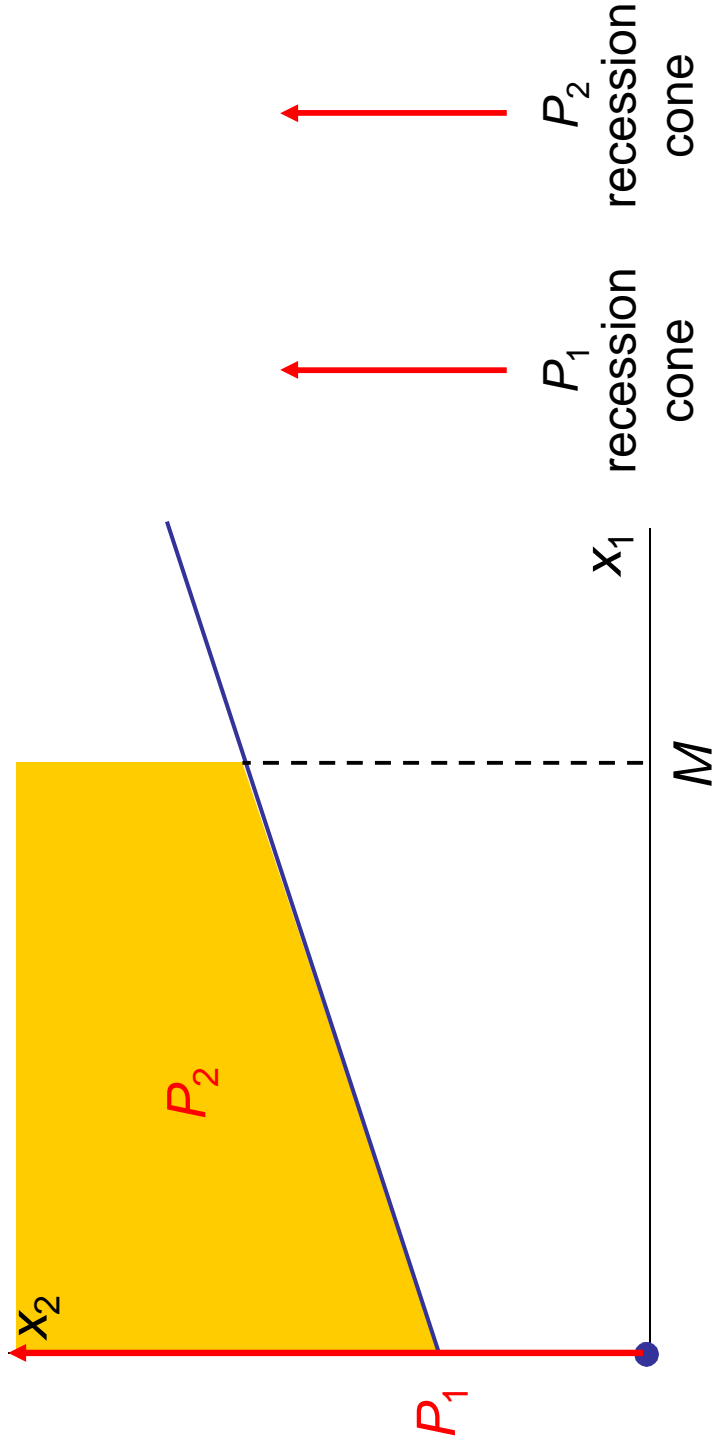
To fix the problem...

$$\min x_2$$

$$x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases}$$

$$0 \leq x_1 \leq M$$

Add an upper bound on x_1

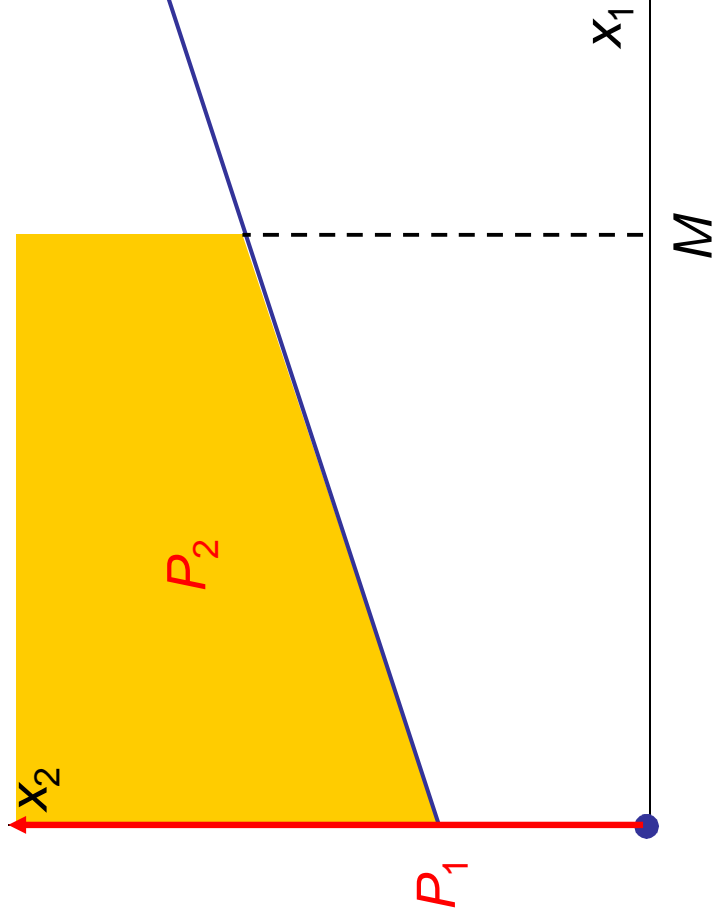


The polyhedra have the same recession cone.

Fixed charge problem

The disjunction is now...

$$\min x_2$$
$$\left(\begin{array}{l} x_1 = 0 \\ x_2 \geq 0 \end{array} \right) \vee \left(\begin{array}{l} 0 \leq x_1 \leq M \\ x_2 \geq f + cx_1 \end{array} \right)$$



Fixed charge problem

$$\min x_2 \quad \vee \begin{pmatrix} x_1 = 0 \\ x_2 \geq 0 \end{pmatrix} \vee \begin{pmatrix} 0 \leq x_1 \leq M \\ x_2 \geq f + cx_1 \end{pmatrix}$$

The disjunctive model is

$$\begin{aligned} \min x_2 \\ x_1^1 = 0 \quad 0 \leq x_1^2 \leq My_2 \\ x_2^1 \geq 0 \quad -cx_1^2 + x_2^2 \geq fy_2 \\ y_1 + y_2 = 1, \quad y_k \in \{0,1\} \\ x_1 = x_1^1 + x_1^2, \quad x_2 = x_2^1 + x_2^2 \end{aligned}$$

Fixed charge problem

$$\min x_2$$
$$\left(\begin{array}{l} x_1 = 0 \\ x_2 \geq 0 \end{array} \right) \vee \left(\begin{array}{l} 0 \leq x_1 \leq M \\ x_2 \geq f + cx_1 \end{array} \right)$$

Simplify the model

$$\min x_2$$
$$0 \leq x_1 \leq My$$
$$x_2 \geq cx_1 + fy$$
$$y \in \{0,1\}$$

Fixed charge problem

$$\min x_2$$
$$\left(\begin{array}{l} x_1 = 0 \\ x_2 \geq 0 \end{array} \right) \vee \left(\begin{array}{l} 0 \leq x_1 \leq M \\ x_2 \geq f + cx_1 \end{array} \right)$$

Simplify the model

$$\min x_2$$
$$0 \leq x_1 \leq My$$
$$x_2 \geq cx_1 + fy$$
$$y \in \{0,1\}$$

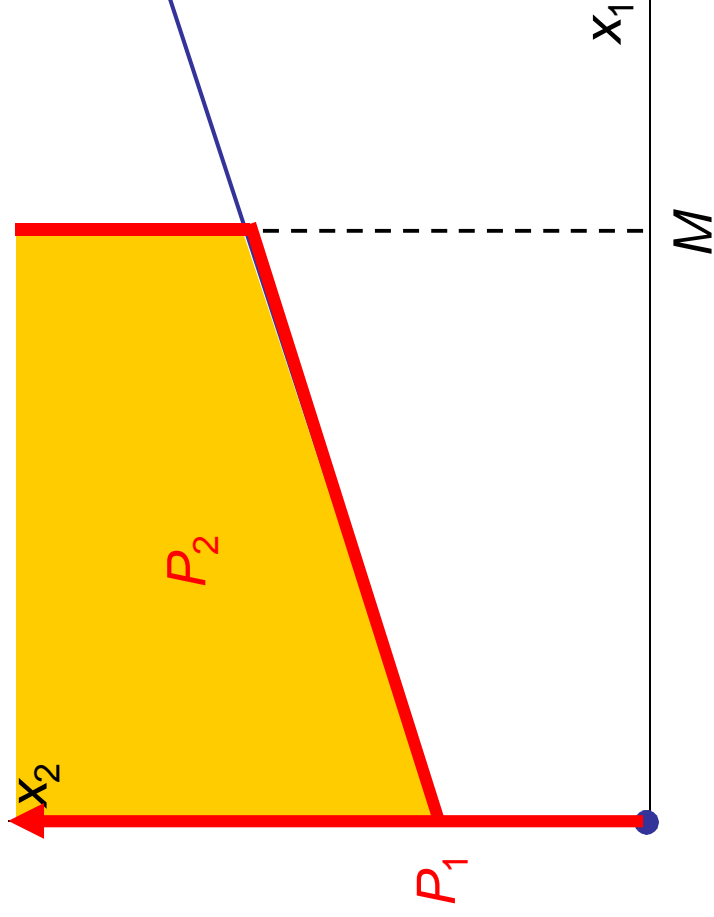
Previous model

$$\min x_2$$
$$x_1 \geq 0$$
$$x_2 \geq cx_1 + fy$$
$$y \in \{0,1\}$$

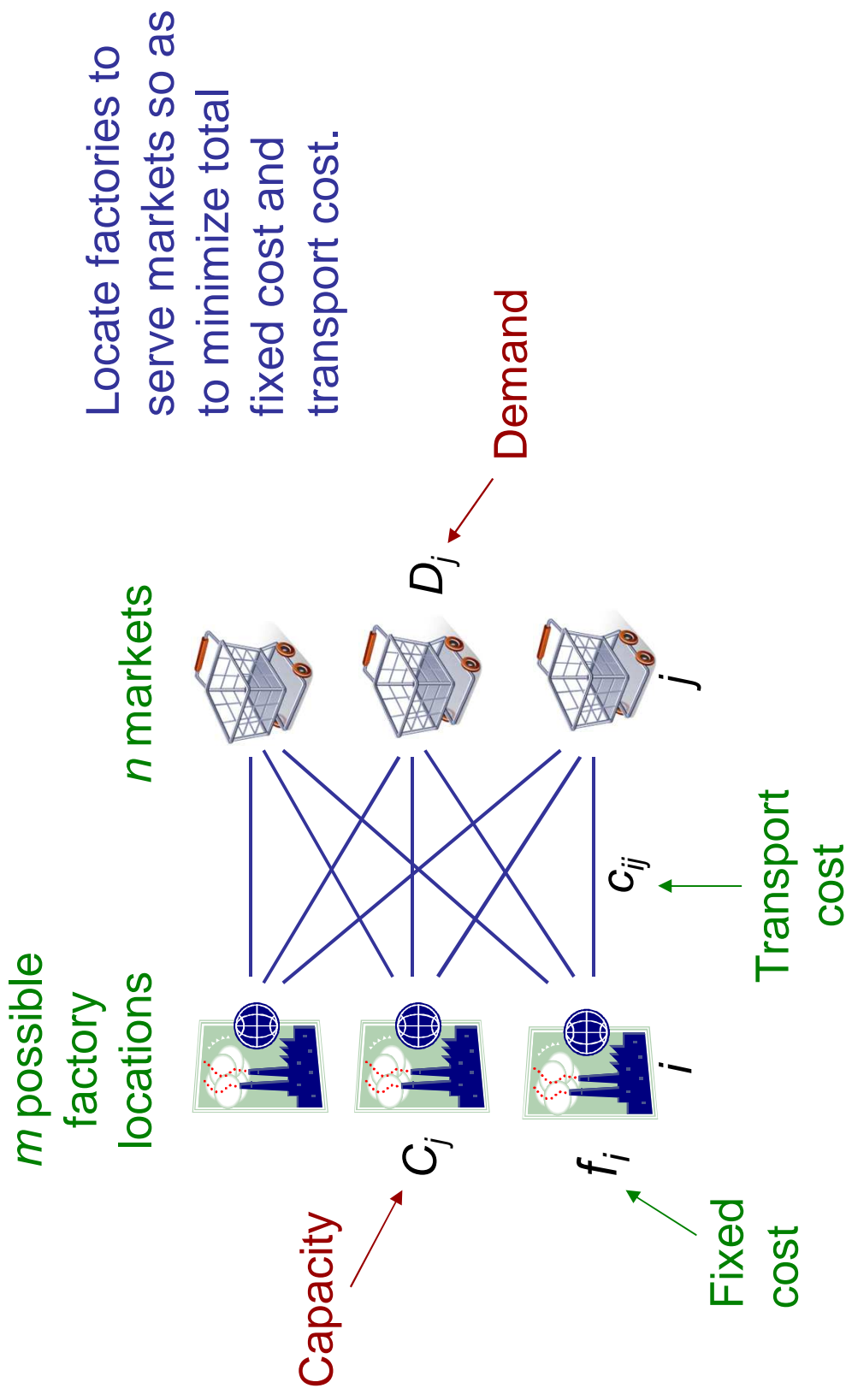
$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & 0 \leq x_1 \leq My \\ & x_2 \geq cx_1 + fy \\ & y \in \{0,1\} \end{aligned}$$

“Big M”

The model now correctly describes the feasible set.



Example: Facility location

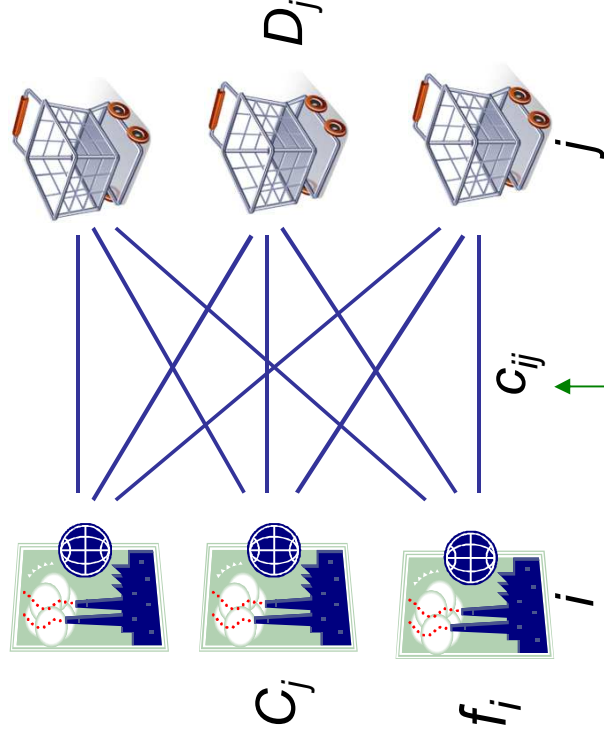


Facility location

m possible
factory
locations

n markets

f_i C_j C_{ij} D_j j



Fixed
cost

Transport
cost

Amount shipped
from factory i to
market j

Disjunctive model:

$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$

$$\left(\begin{array}{l} \sum_j x_{ij} \leq C_i \\ z_i \geq f_i \\ x_{ij} \geq 0, \text{ all } j \end{array} \right) \vee \left(\begin{array}{l} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{array} \right), \text{ all } i$$

$$\sum_i x_{ij} = D_j, \text{ all } j$$

No factory
at location i

Factory at
location i

Facility location



$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$
$$\left(\begin{array}{l} \sum_j x_{ij} \leq C_i \\ z_i \geq f_i \\ x_{ij} \geq 0, \text{ all } j \end{array} \right) \vee \left(\begin{array}{l} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{array} \right), \text{ all } i$$
$$\sum_i x_{ij} = D_j, \text{ all } j$$

Disjunctive model:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

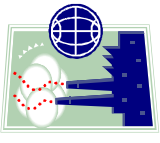
$$\sum_j x_{ij} \leq C_i y_i, \text{ all } i$$

$$\sum_i x_{ij} = D_j, \text{ all } j$$

$$y_i \in \{0,1\}, \quad x_{ij} \geq 0, \text{ all } i, j$$

MILP formulation:

Uncapacitated facility location



Beginner's mistake: Model it as special case of capacitated problem

$$\begin{aligned} \min \quad & \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij} \\ \sum_j x_{ij} & \leq n y_i, \text{ all } i \\ y_i & \in \{0, 1\} \\ \sum_i x_{ij} & = 1, \text{ all } j \end{aligned}$$

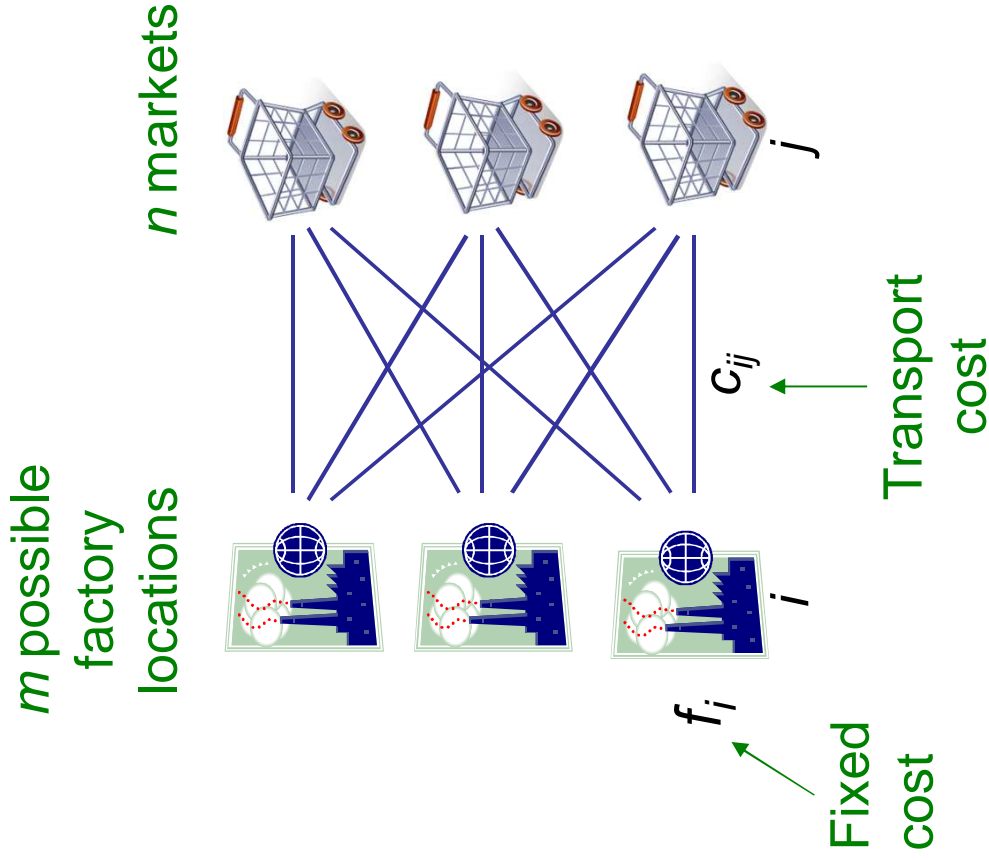
Fraction of demand j satisfied by factory i

Factory i has max output n

This is not the best model.

We can obtain a tighter model by starting with disjunctive formulation.

Uncapacitated facility location



Fraction of demand j satisfied by factory i

Disjunctive model:

$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$

$$\left(\begin{array}{l} 0 \leq x_{ij} \leq 1, \text{ all } j \\ z_i \geq f_i \end{array} \right) \vee \left(\begin{array}{l} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{array} \right), \text{ all } i$$

$$\sum_i x_{ij} = 1, \text{ all } j$$

No factory at location i

Factory at location i

Uncapacitated facility location



$$\text{MILP formulation: } \min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

$$0 \leq x_{ij} \leq y_i, \text{ all } i, j$$

$$\sum_j x_{ij} = 1, \text{ all } j$$

$$y_i \in \{0, 1\}, \text{ all } i$$

This is the textbook model.

More constraints, but tighter relaxation.

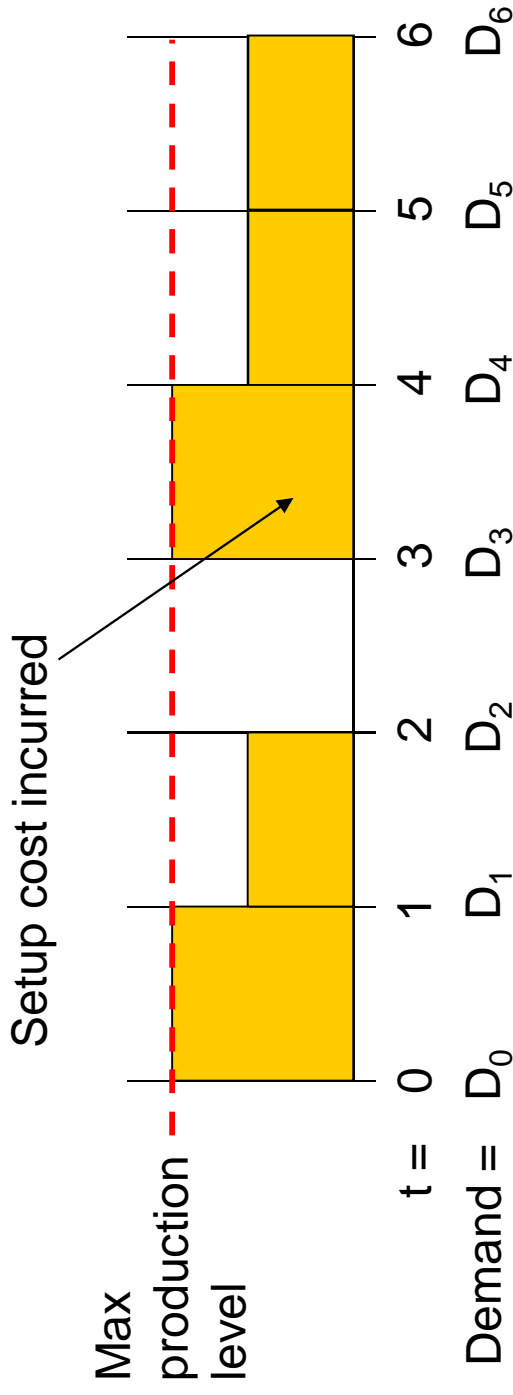
$$\text{Beginner's model: } \min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

$$\sum_j x_{ij} \leq n y_i, \text{ all } i$$

$$\sum_i x_{ij} = 1, \text{ all } j$$

$$y_i \in \{0, 1\}, \text{ all } i$$

Example: Lot sizing with setup costs



Determine lot size in each period to minimize total production, inventory, and setup costs.

$$\begin{array}{c}
 \text{Fixed-cost} \\
 \text{variable}
 \end{array}
 \left(\begin{array}{l}
 V_t \geq f_t \\
 0 \leq x_t \leq C_t
 \end{array} \right) \vee
 \begin{array}{c}
 \text{Fixed} \\
 \text{cost}
 \end{array}
 \left(\begin{array}{l}
 V_t \geq 0 \\
 0 \leq x_t \leq C_t
 \end{array} \right) \vee
 \begin{array}{c}
 \text{Production} \\
 \text{capacity}
 \end{array}
 \left(\begin{array}{l}
 V_t \geq 0 \\
 0 \leq x_t \leq C_t
 \end{array} \right) \vee
 \begin{array}{c}
 \text{Production} \\
 \text{level}
 \end{array}
 \left(\begin{array}{l}
 V_t \geq 0 \\
 x_t = 0
 \end{array} \right)$$

(1)

(2)

(3)

Start production (incurs setup cost)	Continue production (no setup cost)	Produce nothing (no production cost)
-----------------------------------------------	----------------------------------------------	-----------------------------------------------

Logical conditions:

(2) In period $t \Rightarrow$ (1) or (2) in period $t - 1$

(1) In period $t \Rightarrow$ neither (1) nor (2) in period $t - 1$

(1) (2) (3)

Start production Continue production Produce nothing

$$\left(\begin{array}{l} v_t \geq f_t \\ 0 \leq x_t \leq C_t \end{array} \right) \vee \left(\begin{array}{l} v_t \geq 0 \\ 0 \leq x_t \leq C_t \end{array} \right) \vee \left(\begin{array}{l} v_t \geq 0_t \\ x_t = 0 \end{array} \right)$$

Convex hull MILP model of disjunction:

$$\begin{array}{lll} v_t^1 \geq f_t y_{t1} & v_t^2 \geq 0 & v_t^3 \geq 0 \\ 0 \leq x_t^1 \leq C_t y_{t1} & 0 \leq x_t^2 \leq C_t y_{t2} & x_t^3 = 0 \end{array}$$

$$v_t = \sum_{k=1}^3 v_t^k, \quad x_t = \sum_{k=1}^3 x_t^k, \quad y_t = \sum_{k=1}^3 y_{tk}$$

$$y_{tk} \in \{0,1\}, \quad k = 1,2,3$$

(1) (2) (3)

Start production Continue production Produce nothing

$$\left(\begin{array}{l} v_t \geq f_t \\ 0 \leq x_t \leq C_t \end{array} \right) \vee \left(\begin{array}{l} v_t \geq 0 \\ 0 \leq x_t \leq C_t \end{array} \right) \vee \left(\begin{array}{l} v_t \geq 0_t \\ x_t = 0 \end{array} \right)$$

Simplify the MILP model:

$$\begin{aligned} v_t &\geq f_t z_t \\ 0 \leq x_t &\leq C_t(z_t + y_t) \\ z_t + y_t &\leq 1 \\ z_t, y_t &\in \{0,1\}, \quad k = 1,2,3 \end{aligned}$$

Formulate logical conditions:

(2) In period $t \Rightarrow$ (1) or (2) in period $t - 1$

(1) In period $t \Rightarrow$ neither (1) nor (2) in period $t - 1$

$$v_t \geq f_t z_t$$

$$0 \leq x_t \leq C_t(z_t + y_t)$$

$$z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

$$y_t \leq z_{t-1} + y_{t-1}$$

$$z_t \leq 1 - z_{t-1} - y_{t-1}$$

Add objective function

Unit production cost Unit holding cost

$$\min \sum_{t=1}^n (p_t x_t + h_t s_t + v_t)$$

$$v_t \geq f_t z_t$$

$$0 \leq x_t \leq C_t(z_t + y_t)$$

$$z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

$$y_t \leq z_{t-1} + y_{t-1}$$

$$z_t \leq 1 - z_{t-1} - y_{t-1}$$

Big-M Disjunctive Formulation

Again start with a disjunction of linear systems.

$$\bigvee_k (A^k x \geq b^k)$$

y_k is 1 when x is in polyhedron k .

$$A^k x^k \geq b^k - (1 - y_k) M^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$y_k \in \{0, 1\}, \text{ all } k$$

M^k is a vector of bounds that makes system k nonbinding when $y_k = 0$.

$$M^k = b^k - \min \left\{ A^k x \mid \bigvee_{\ell \neq k} (A^\ell x \geq b^\ell) \right\}$$

Big M



Big-M Disjunctive Formulation

Again start with a disjunction of linear systems.

$$\bigvee_k (A^k x \geq b^k)$$

y_k is 1 when x is in polyhedron k .

$$A^k x \geq b^k - (1 - y_k) M^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$y_k \in \{0, 1\}, \text{ all } k$$

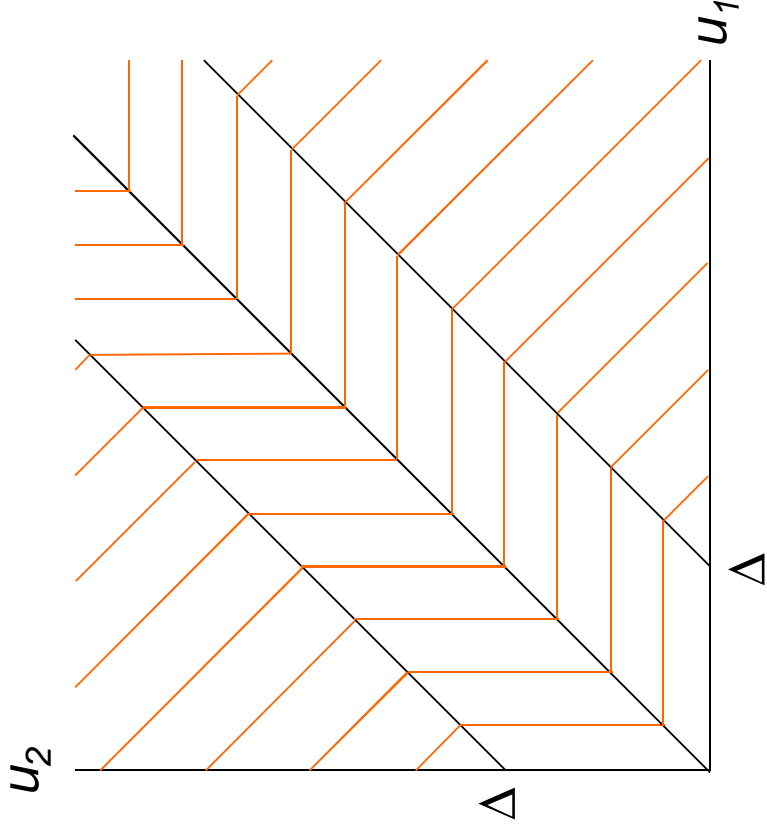
M^k is a vector of bounds that makes system k nonbinding when $y_k = 0$.

$$M^k = b^k - \min \left\{ A^k x \mid \bigvee_{\ell \neq k} (A^\ell x \geq b^\ell) \right\}$$

Every bounded MILP-representable set has a model of this form (as well as a convex hull disjunctive model).

Example: Health Care Benefits

Distribute limited health benefits to two persons.
Person i receives utility u_i .



Two criteria:

If $|u_1 - u_2| \leq \Delta$, Rawlsian:

$$\max \min\{u_1, u_2\}$$

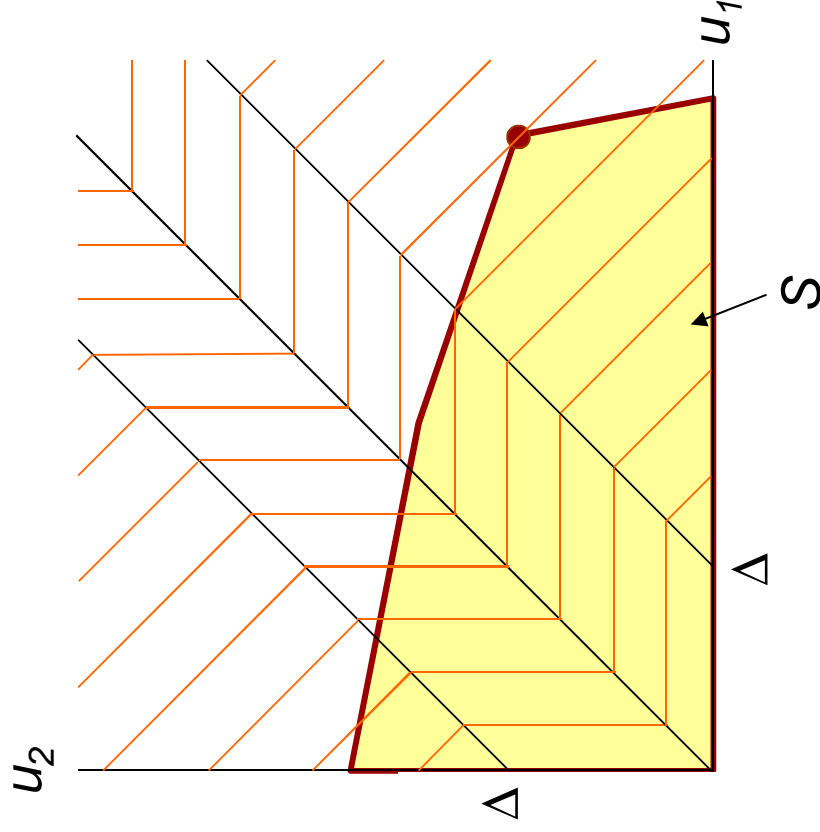
If $|u_1 - u_2| > \Delta$, utilitarian:

$$\max u_1 + u_2$$

Maximize welfare of person who is more seriously ill, unless this requires too much sacrifice from the other person.

Example: Health Care Benefits

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Optimization problem:

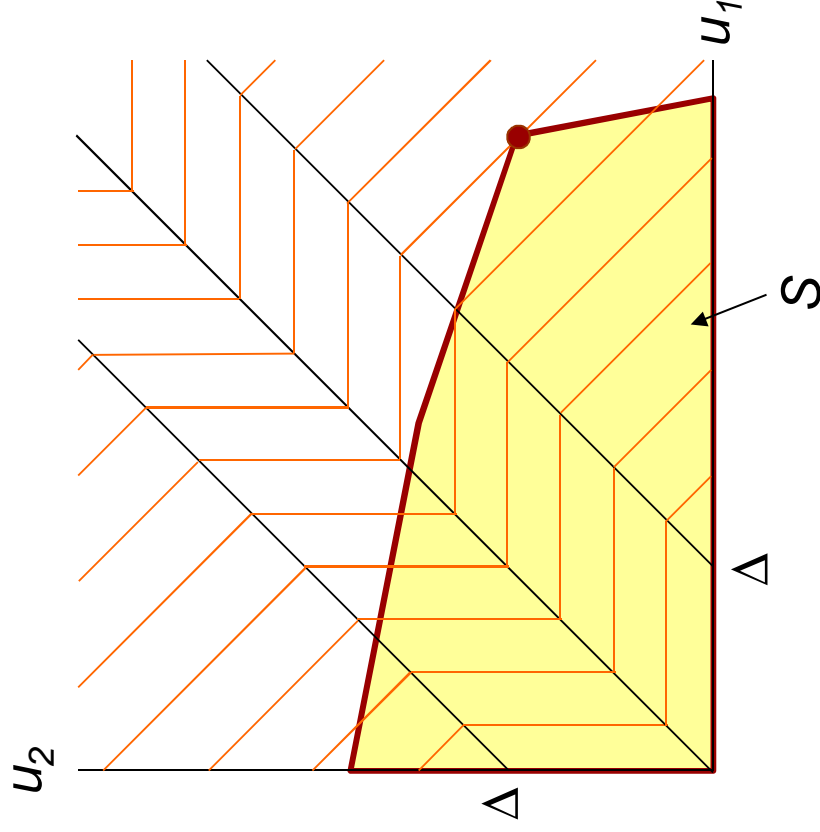
$\max z$

$$z \leq \begin{cases} 2 \min\{u_1, u_2\} + \Delta & \text{if } |u_1 - u_2| \leq \Delta \\ u_1 + u_2 & \text{otherwise} \end{cases}$$

$$u_1, u_2 \in S$$

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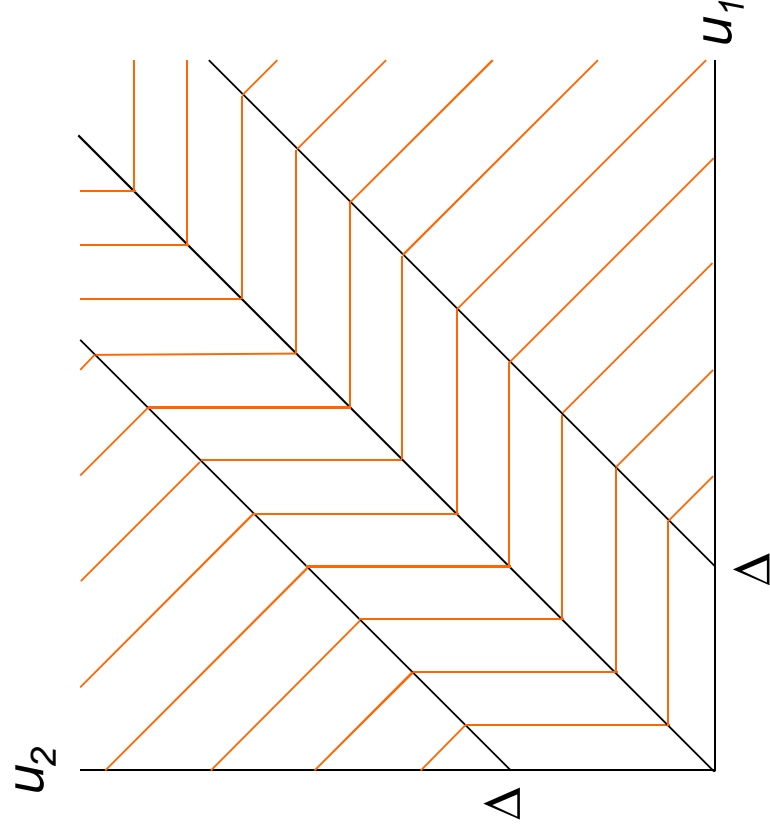
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$$u_1, u_2 \in S$$

Ensures continuity

Example: Health Care Benefits

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Ignoring S , we would like a convex hull MILP model of the epigraph.

Can we do it?

No!

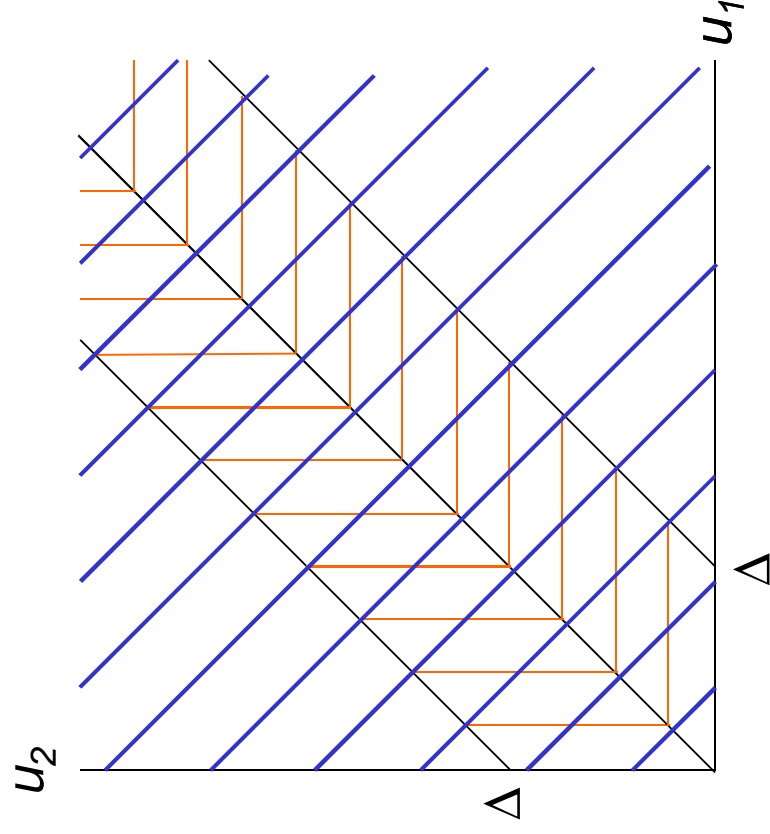
Optimization problem:

$$\begin{aligned} \max z \\ z \leq & \begin{cases} 2 \min\{u_1, u_2\} + \Delta & \text{if } |u_1 - u_2| \leq \Delta \\ u_1 + u_2 & \text{otherwise} \end{cases} \\ & u_1, u_2 \in S \end{aligned}$$

Example: Health Care Benefits

Distribute limited health benefits to two persons.

Person i receives utility u_i .



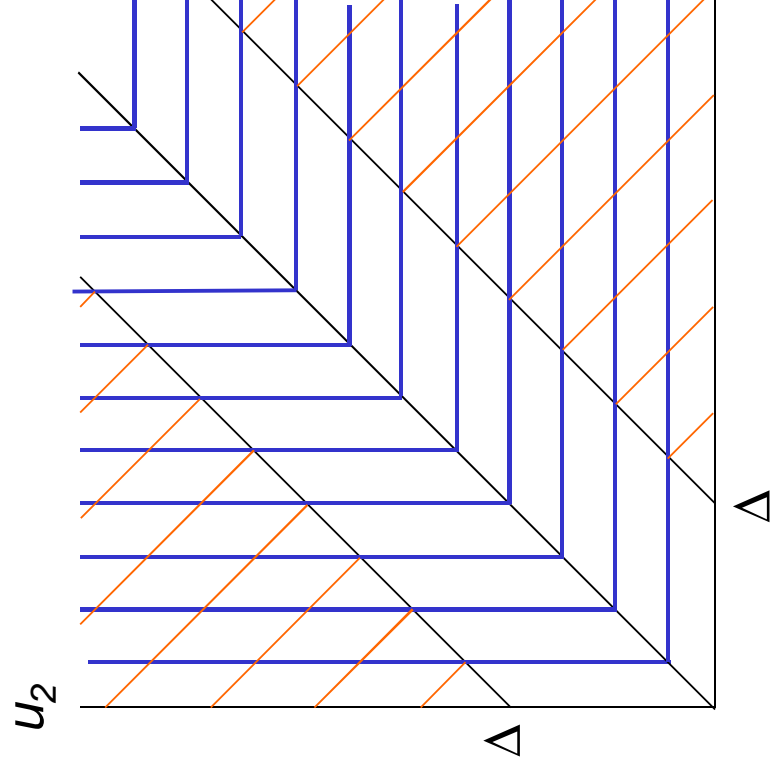
Epigraph is union of two polyhedra:

P_1 has recession cone $\{(\alpha, \beta, z) \mid z \leq \alpha + \beta, \alpha, \beta \geq 0\}$

Example: Health Care Benefits

Distribute limited health benefits to two persons.

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Epigraph is union of two polyhedra:

P_1 has recession cone

$$\{(\alpha, \beta, z) \mid z \leq \alpha + \beta, \alpha, \beta \geq 0\}$$

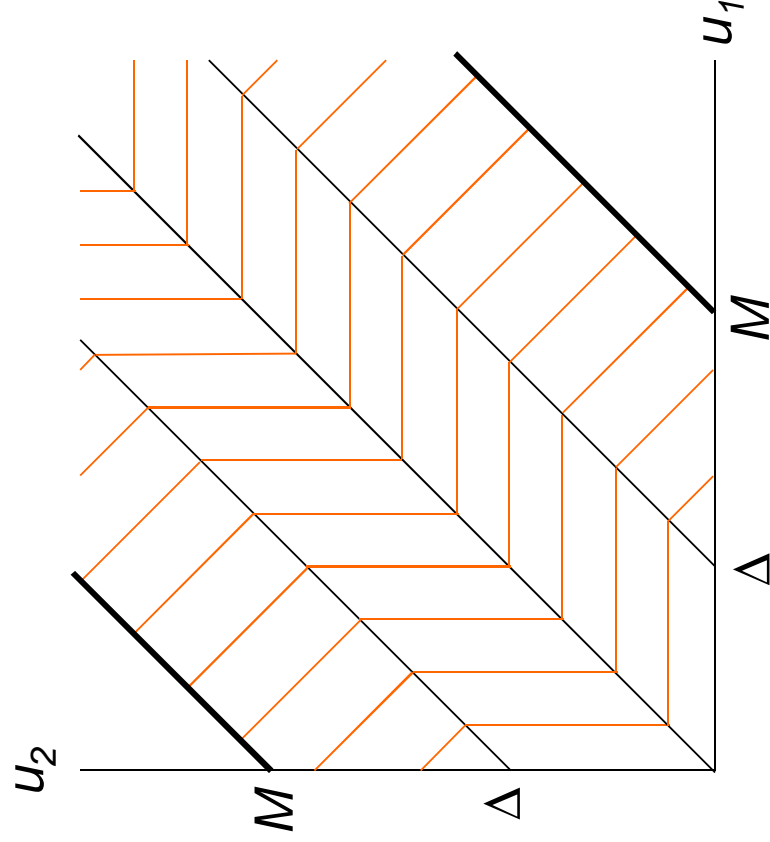
P_2 has recession cone

$$\{(1, 1, z) \mid 0 \leq z \leq 2\} \cup \{(1, 0, 0), (0, 1, 0)\}$$

Example: Health Care Benefits

Distribute limited health benefits to two persons.

Person i receives utility u_i .



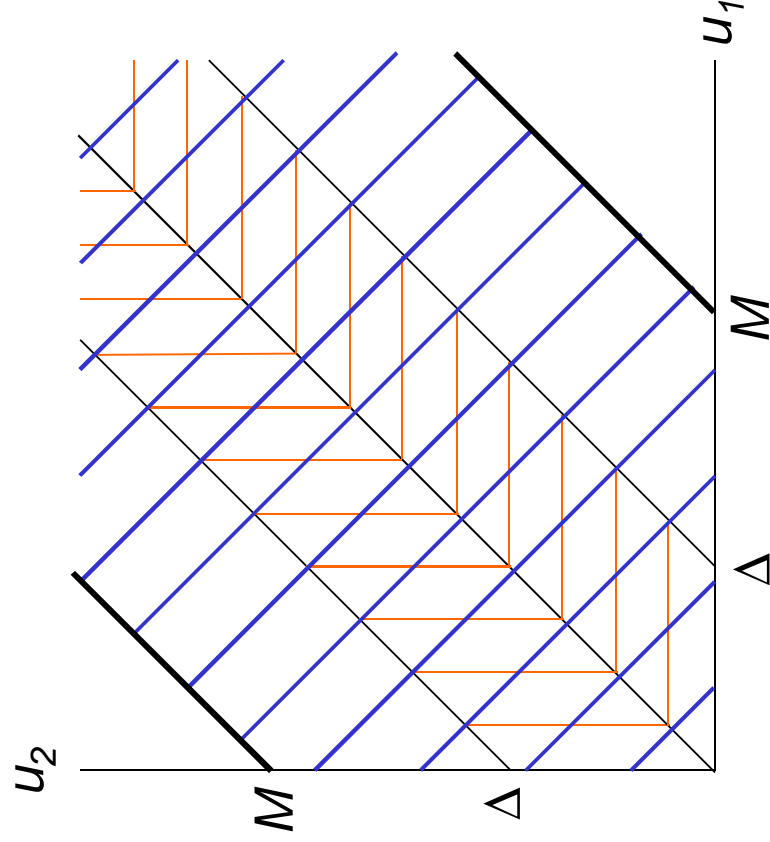
Solution:

Add constraint $|u_1 - u_2| \leq M$

No need to bound u_1, u_2 individually

Example: Health Care Benefits

Distribute limited health benefits to two persons.
Person i receives utility u_i .



Solution:

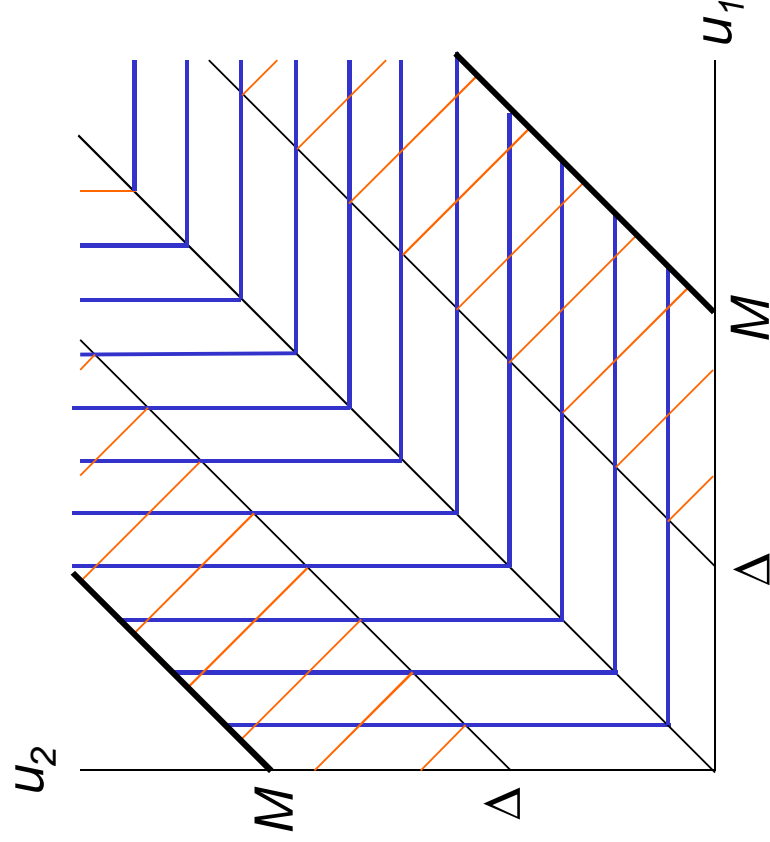
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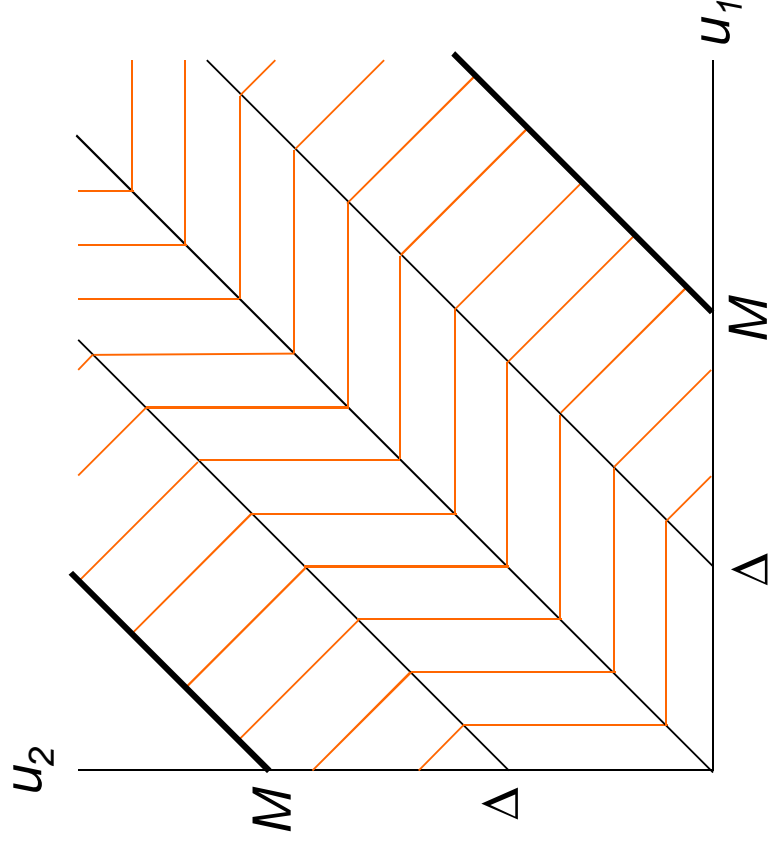
P_1 has recession cone $\{(1,1,z) \mid 0 \leq z \leq 2\}$

So does P_2

Example: Health Care Benefits

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Person i receives utility u_i .



Big-M model:

$$z \leq 2u_1 + \Delta + (M - \Delta)y$$

$$z \leq 2u_2 + \Delta + (M - \Delta)y$$

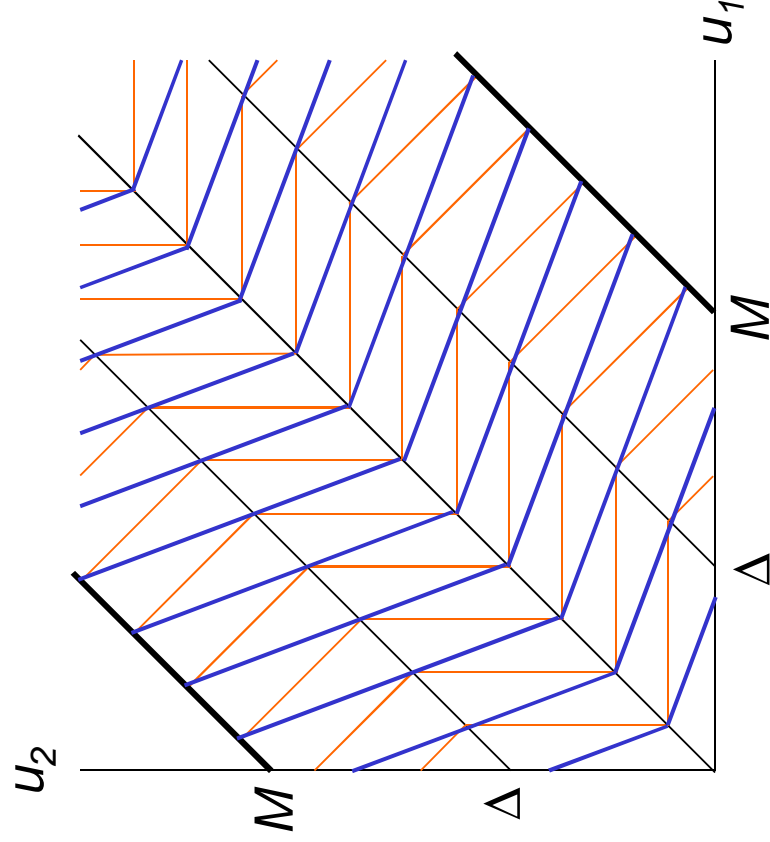
$$z \leq u_1 + u_2 + \Delta(1 - y)$$

$$u_1 - u_2 \leq M, \quad u_2 - u_1 \leq M$$

$$u_1, u_2 \geq 0, \quad y \in \{0, 1\}$$

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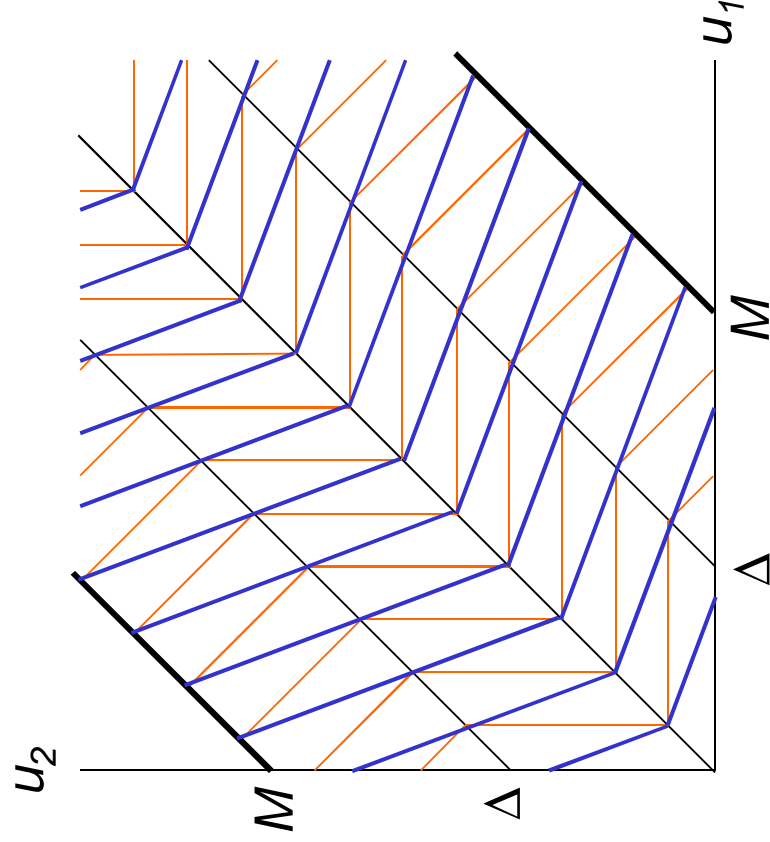
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$$u_1, u_2 \geq 0, \quad y \in \{0, 1\}$$

Theorem: This is a convex hull formulation.

Model is no tighter if we use $u_1, u_2 \leq M$

Example: Health Care Benefits

Optimization problem for the n -person case:

max z

$$z \leq (n-1)\Delta + n \min_j \{u_j\} + \sum_{j=1}^n \max \{0, u_j - \min_j \{u_j\} - \Delta\}$$

$$|u_i - u_j| \leq M, \text{ all } i, j$$

$$u \geq 0, \quad u \in S$$

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Big- M disjunctive model:

max z

$$z \leq -\Delta + \sum_{j=1}^n w_{ij}, \text{ all } i$$

$$w_{ij} \leq \Delta + u_i + y_{ij}(M - \Delta), \text{ all } i, j$$

$$w_{ij} \leq u_j + (1 - y_{ij})\Delta, \text{ all } i, j$$

$$y_{ij} = 0, \text{ all } i$$

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Theorem:
This is a convex hull formulation.

General MILP Representability

Knapsack Models

- Integer variables can also be used to express counting ideas.
- This is totally different from the use of 0-1 variables to express unions of polyhedra.
- Examples:
 - Knapsack inequalities
 - Packing and covering
 - Logical clauses
 - Cost bounds



Knapsack Models

- Disjunctive representability does not accommodate knapsack constraints in a natural way.
- Knapsack constraints are bounded MILP representable only if integer variables are bounded.
- ...and only in a technical sense.
- By regarding each integer lattice point as a polyhedron.



General representability theorem

Integer variables can now be **unbounded**:

A subset S of $\mathbb{R}^n \times \mathbb{Z}^p$ is **MILP representable** if S is the projection onto x of the feasible set of some MILP constraint set of the form

$$Ax + Bz + Dy \geq b$$

$x \in \mathbb{R}^n \times \mathbb{Z}^p$, $z \in \mathbb{R}^m$
 $y \in \{0,1\}^q$

Some modeling variables are continuous, some integer

Auxiliary continuous variables can be used

General representability theorem

Integer variables can be unbounded:

A subset S of $\mathbb{R}^n \times \mathbb{Z}^p$ is **MILP representable** if S is the projection onto x of the feasible set of some MILP constraint set of the form

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Assume that A, B, D, b consist of **rational data**

General representability theorem

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Assume that A, B, D, b consist of **rational data**

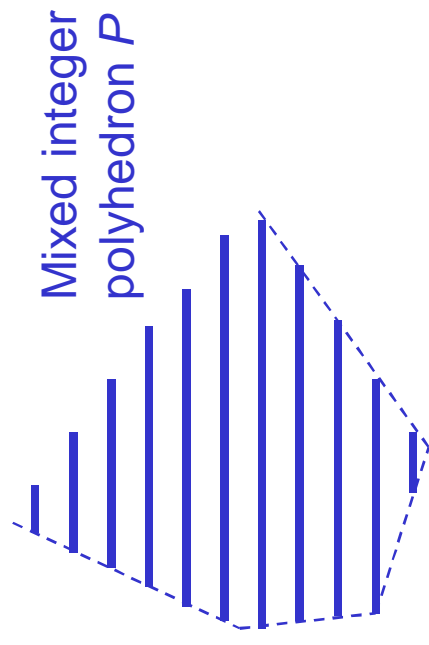
A **mixed integer polyhedron** is any set of the form

$$\{x \in \mathbb{R}^n \times \mathbb{Z}^p \mid Ax \geq b\}$$

General representability theorem

Rational vector d is a **recession direction** of a mixed integer polyhedron $P \subset \mathbb{R}^n \times \mathbb{Z}^p$ if it is a recession direction of some polyhedron $Q \subset \mathbb{R}^{n+p}$ for which

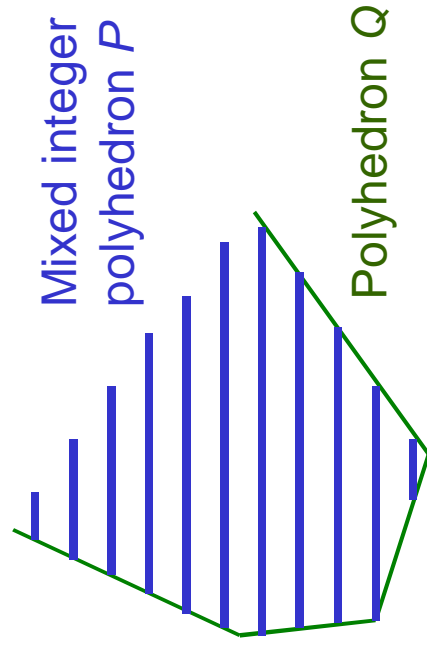
$$P = Q \cap (\mathbb{R}^n \times \mathbb{Z}^p)$$



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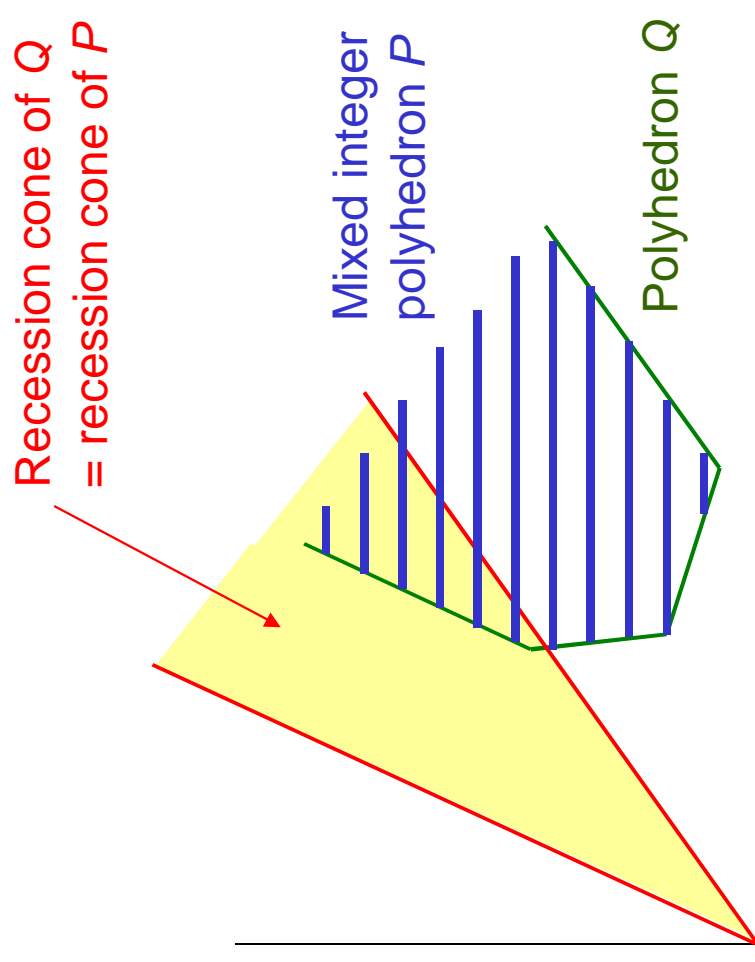
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General representability theorem

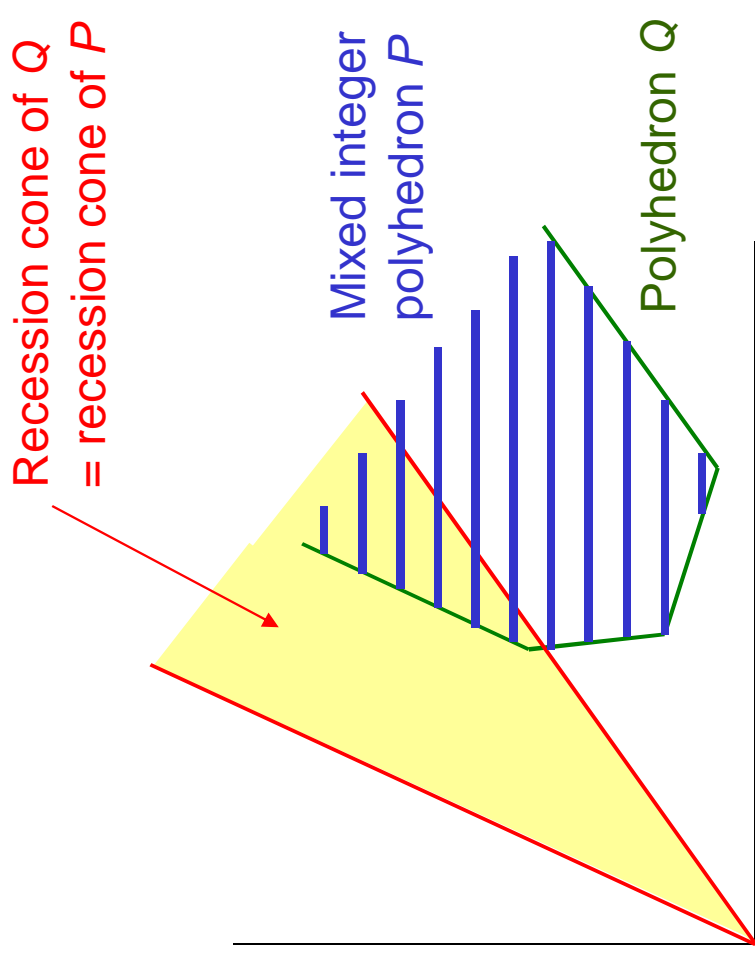
Rational vector d is a **recession direction** of a mixed integer polyhedron $P \subset \mathbb{R}^n \times \mathbb{Z}^p$ if it is a recession direction of some polyhedron $Q \subset \mathbb{R}^{n+p}$ for which

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General representability theorem

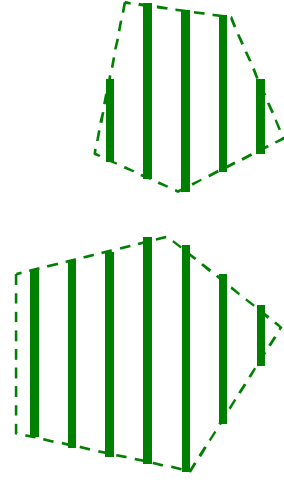
Lemma. All polyhedra in \mathbb{R}^{n+p} having the same **nonempty** intersection with $\mathbb{R}^n \times \mathbb{Z}^p$ have the same recession cone.



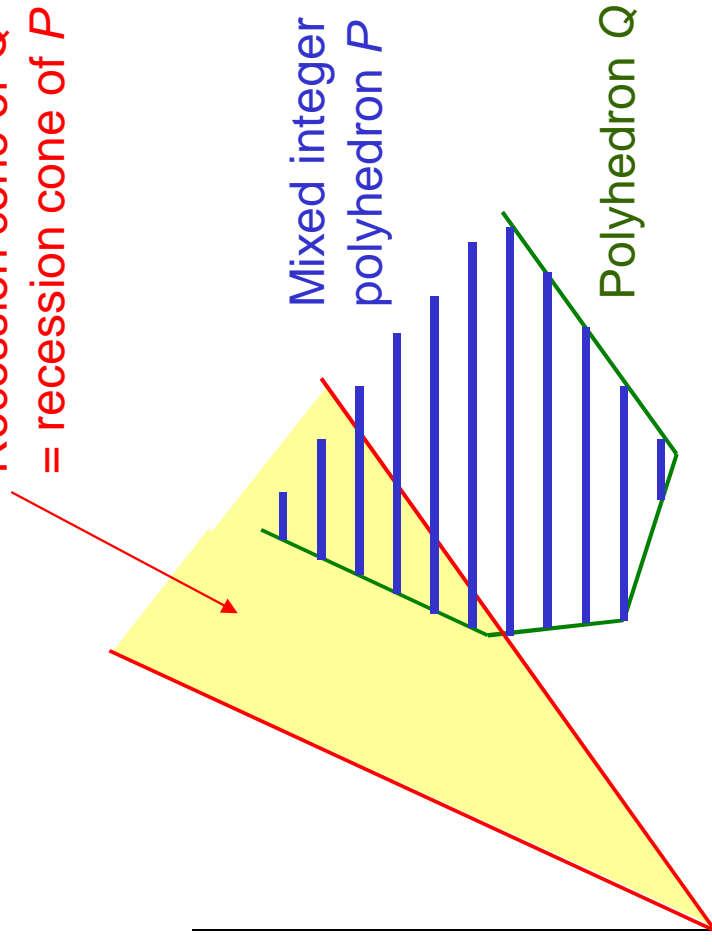
General representability theorem

Theorem. A nonempty subset of $\mathbb{R}^n \times \mathbb{Z}^p$ is MILP representable if and only if it is the union of finitely many mixed integer polyhedra in $\mathbb{R}^n \times \mathbb{Z}^p$ having the same recession cone.

Recession cone of Q
= recession cone of P



Union of mixed integer polyhedra with the same recession cone (in this case, the origin)



Convex Hull Formulation

Start with a disjunction of linear systems to represent the union of mixed integer polyhedra.

$$\bigvee_k (A^k x \geq b^k)$$

The k th polyhedron is $\{x \in \mathbb{R}^n \times \mathbb{Z}^p \mid A^k x \geq b^k\}$

Aside from domain of x , the disjunctive model is the same as before.

$$A^k x^k \geq b^k y_k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k x^k$$

$$x \in \mathbb{R}^n \times \mathbb{Z}^p, y_k \in \{0,1\}$$

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Every MILP representable set has a model of this form.

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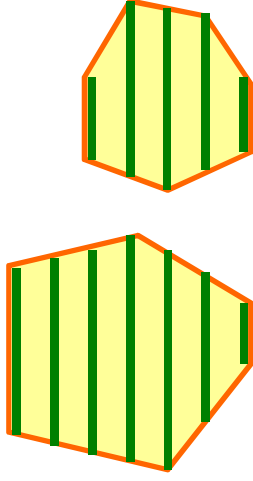
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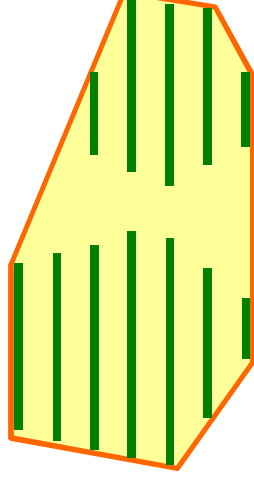
...also a model in disjunctive big-M form.

Convex Hull Formulation

Theorem. If each mixed integer polyhedron has a convex hull formulation $A^k x \geq b^k$, the disjunctive model is a **convex hull** formulation of the disjunction.

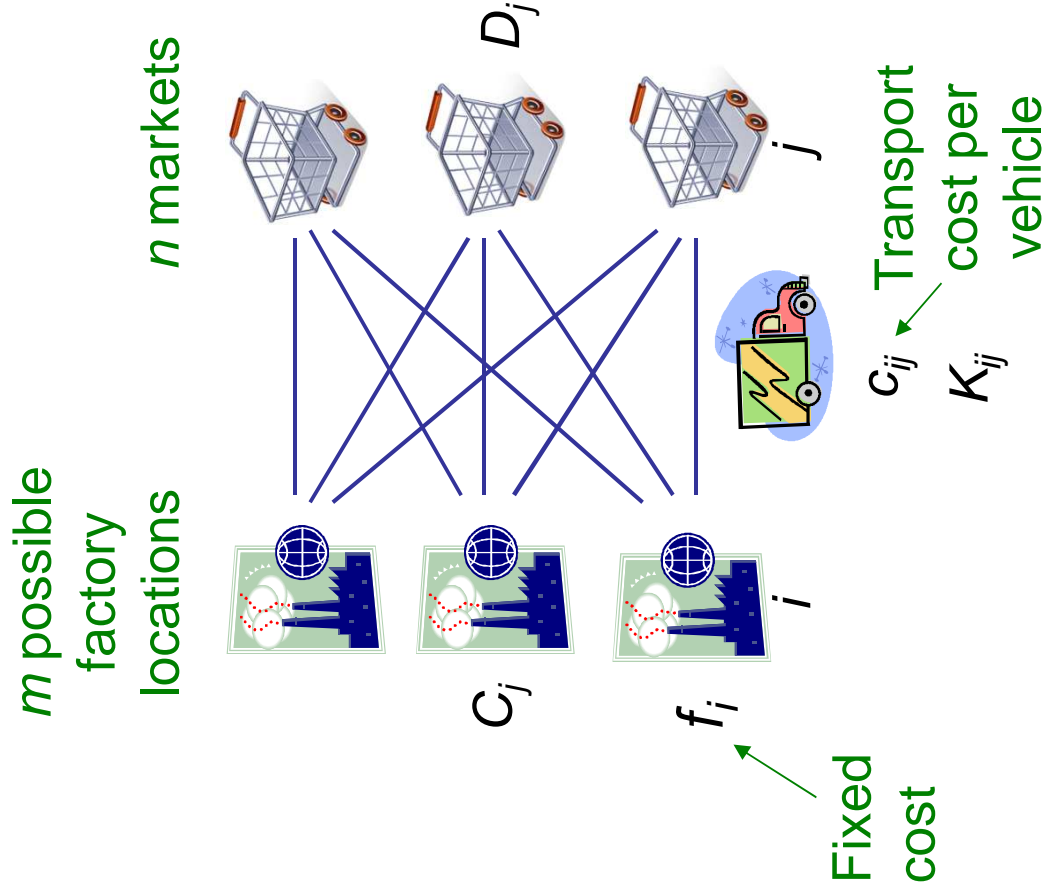


Union of mixed integer polyhedra with convex hull descriptions



Convex hull relaxation

Example: Facility location

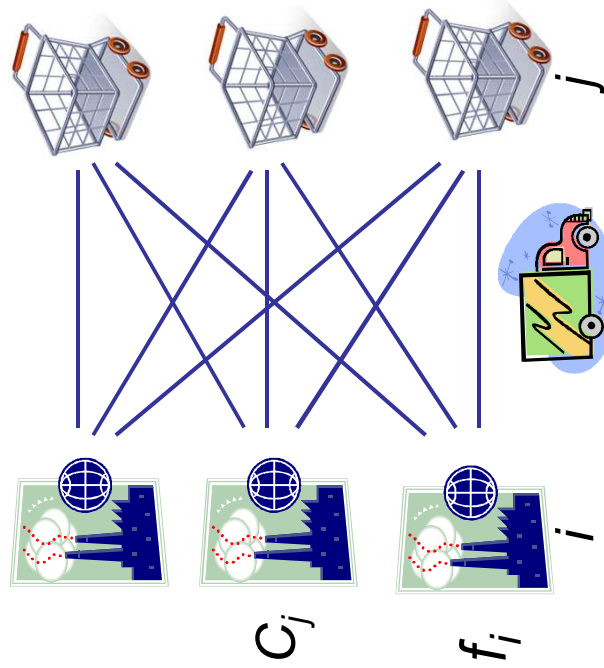


Locate factories to serve markets so as to minimize total factory cost and transport cost.

Fixed cost incurred for each vehicle used.

Facility location

m possible factory locations n markets



Fixed cost f_i

C_{ij} Transport cost per vehicle
 K_{ij} cost per vehicle

Number of vehicles from factory i to market j

Disjunctive model:

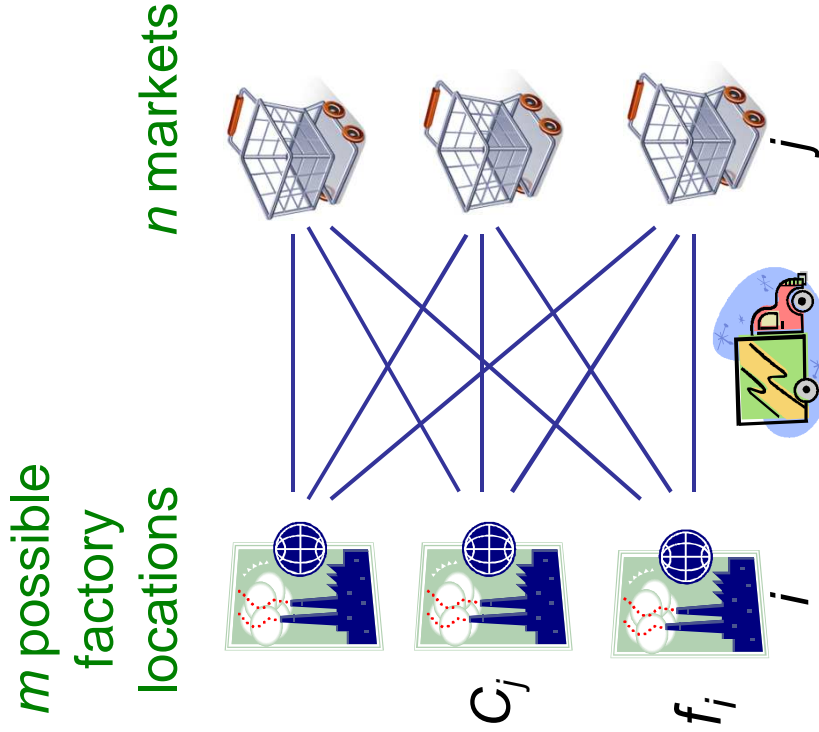
$$\min \sum_i z_i + \sum_{ij} c_{ij} w_{ij}$$

$$\left(\begin{array}{l} \sum_j x_{ij} \leq C_i \\ 0 \leq x_{ij} \leq K_{ij} w_{ij}, \text{ all } j \\ z_i \geq f \\ w_{ij} \in \mathbb{Z}, \text{ all } j \\ \sum_i x_{ij} = D_j, \text{ all } j \end{array} \right) \vee \left(\begin{array}{l} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{array} \right), \text{ all } i$$

No factory at location i

Factory at location i

Facility location



Fixed cost

C_{ij} Transport cost per vehicle

K_{ij}

Disjunctive model:

$$\min \sum_i z_i + \sum_{ij} c_{ij} w_{ij}$$

$$\left\{ \begin{array}{l} \sum_j x_{ij} \leq C_i \\ 0 \leq x_{ij} \leq K_{ij} w_{ij}, \text{ all } j \\ z_i \geq f_i \\ w_{ij} \in \mathbb{Z}, \text{ all } j \end{array} \right.$$

$$\vee \left(\begin{array}{l} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{array} \right), \text{ all } i$$

Number of vehicles from factory i to market j

Describes mixed integer polyhedron

No factory at location i

Factory at location i

Facility location



$$\min \sum_i z_i + \sum_{ij} c_{ij} w_{ij}$$

$$\left(\begin{array}{l} \sum_j x_{ij} \leq C_i \\ 0 \leq x_{ij} \leq K_{ij} w_{ij}, \text{ all } j \\ z_i \geq f \\ w_{ij} \in \mathbb{Z}, \text{ all } j \\ \sum_i x_{ij} = D_j, \text{ all } j \end{array} \right) \vee \left(\begin{array}{l} x_{ij} = 0, \text{ all } j \\ z_i = 0 \\ w_{ij} \geq 0, \text{ all } j \end{array} \right), \text{ all } i$$

$w_{ij} \geq 0$ added to equalize recession cones

Disjunctive model:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} w_{ij}$$

$$\sum_j x_{ij} \leq C_i y_i, \text{ all } i$$

$$\sum_i x_{ij} = D_j, \text{ all } j$$

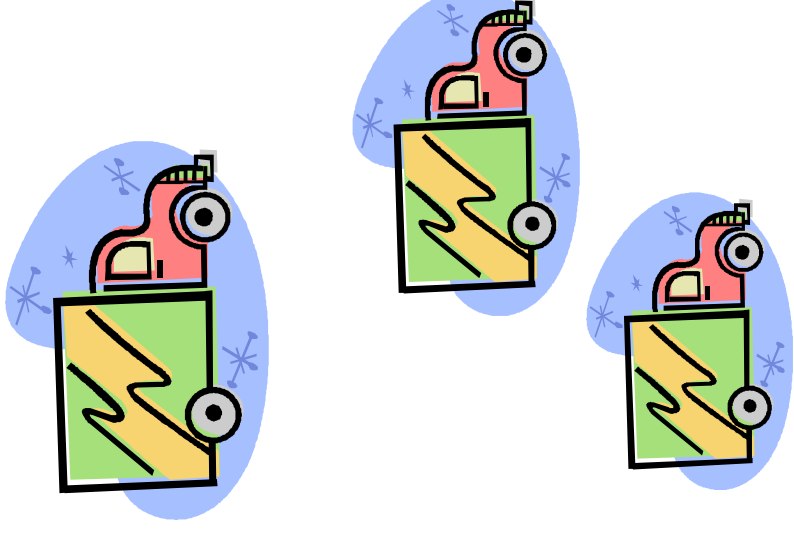
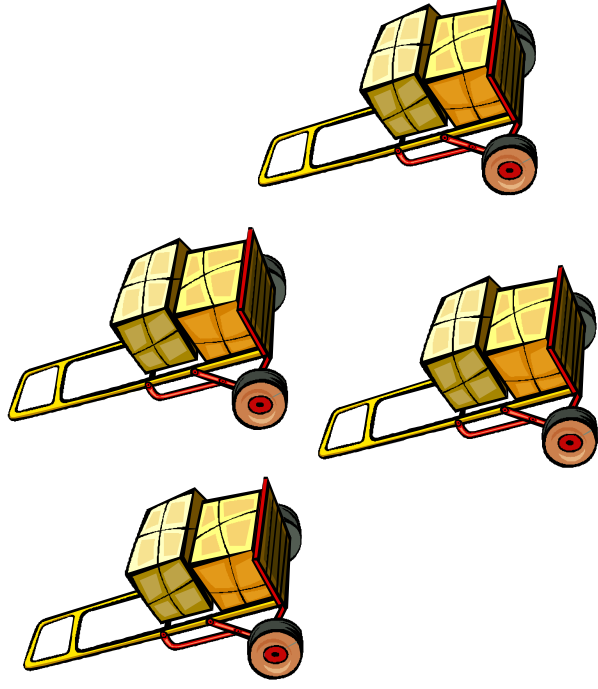
$$0 \leq x_{ij} \leq K_{ij} w_{ij}, \text{ all } i, j$$

$$y_i \in \{0, 1\}, w_{ij} \in \mathbb{Z}, \text{ all } i, j$$

MILP formulation:

Example: Freight Packing and Transfer

- Transport packages using n trucks
- Each package j has size a_j .
- Each truck i has capacity Q_i .



Knapsack component

The trucks selected must have enough capacity to carry the load.

$$\sum_{i=1}^n Q_i y_i \geq \sum_j a_j$$

= 1 if truck i is selected

Disjunctive component

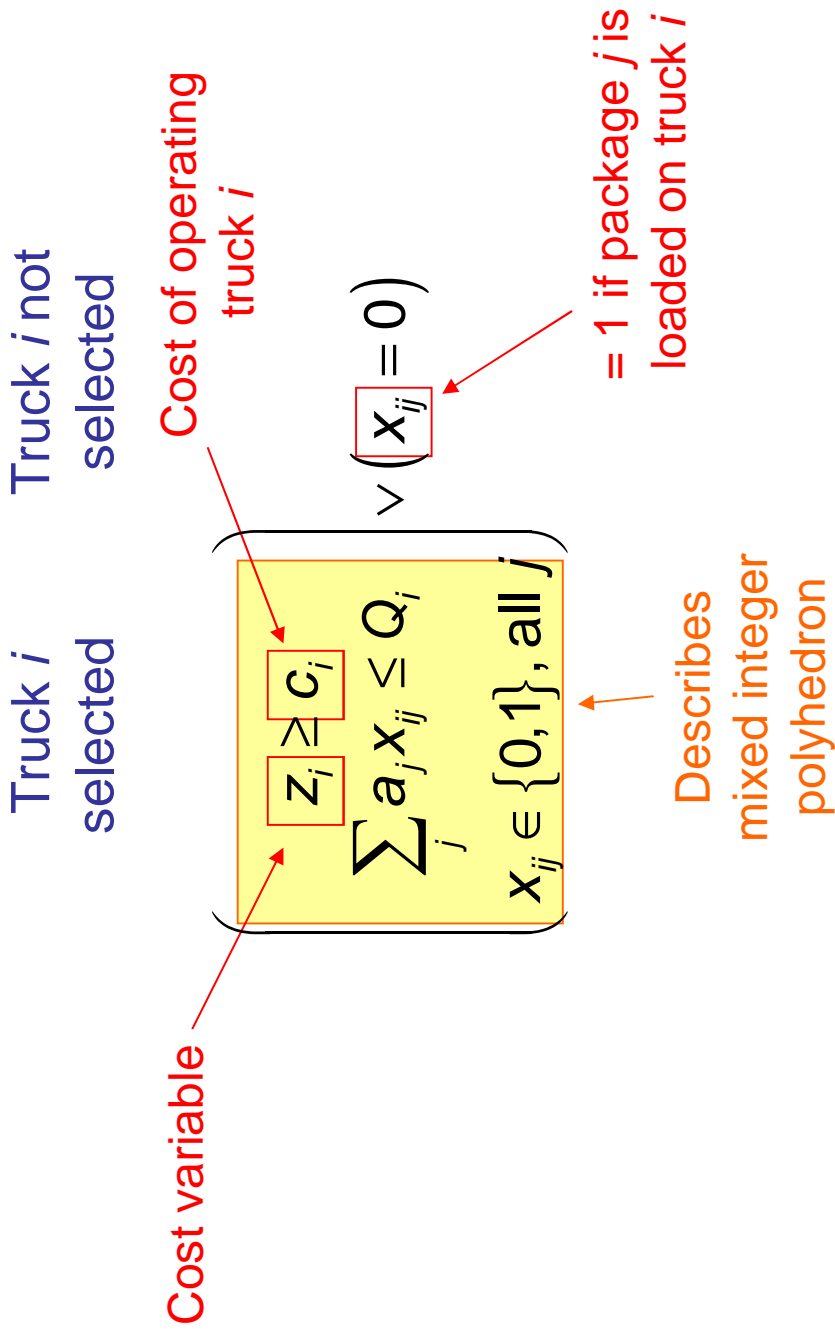
Truck i selected Truck i not selected

Cost variable Cost of operating truck i

$$\left(\begin{array}{l} z_i \geq c_i \\ \sum_j a_j x_{ij} \leq Q_i \\ x_{ij} \in \{0,1\}, \text{ all } j \end{array} \right) \vee (x_{ij} = 0)$$

= 1 if package j is loaded on truck i

Disjunctive component



Disjunctive component

Problem: Recession cones are not the same.

$$\left(\begin{array}{l} z_i \geq c_i \\ \sum_j a_j x_{ij} \leq Q_i \\ x_{ij} \in \{0,1\}, \text{ all } j \end{array} \right) \vee (x_{ij} = 0)$$

$$\{(x_i, z_i) \mid x_i = 0, z_i \geq 0\} \quad \{(x_i, z_i) \mid x_i = 0\}$$

Disjunctive component

Solution: Add innocuous constraint $z_i \geq 0$

$$\left(\begin{array}{l} z_i \geq c_i \\ \sum_j a_j x_{ij} \leq Q_i \\ x_{ij} \in \{0,1\}, \text{ all } j \end{array} \right) \vee \left(\begin{array}{l} z_i \geq 0 \\ x_{ij} = 0 \end{array} \right)$$

$$\{(x_i, z_i) \mid x_i = 0, z_i \geq 0\} \vee \{(x_i, z_i) \mid x_i = 0, z_i \geq 0\}$$

Disjunctive component

Truck i
selected

Truck i not
selected

$$\left(\begin{array}{l} z_i \geq c_i \\ \sum_j a_j x_{ij} \leq Q_i \\ x_{ij} \in \{0,1\}, \text{ all } j \end{array} \right) \vee \left(\begin{array}{l} z_i \geq 0 \\ x_{ij} = 0 \end{array} \right)$$

Convex hull
MILP
formulation

$$z_i \geq c_i y_i$$

$$\sum_j a_j x_{ij} \leq Q_i y_i$$

$$0 \leq x_{ij} \leq y_i$$

The resulting model

$$\min \sum_{i=1}^n c_i y_i$$

$$\sum_j a_j x_{ij} \leq Q_i y_i, \text{ all } i$$
$$0 \leq x_{ij} \leq y_i, \text{ all } i, j$$

Disjunctive
component

$$\sum_{i=1}^n x_{ij} = 1, \text{ all } j$$

Logical condition
(each package must be shipped)

$$\sum_{i=1}^n Q_i y_i \geq \sum_j a_j$$

Knapsack
component

$$x_{ij}, y_i \in \{0,1\}$$

The resulting model

$$\min \sum_{i=1}^n c_i y_i$$

$$\sum_j a_j x_{ij} \leq Q_i y_i, \text{ all } i$$

$$0 \leq x_{ij} \leq y_i, \text{ all } i, j$$

$$\sum_{i=1}^n x_{ij} = 1, \text{ all } j$$

$$\sum_{i=1}^n Q_i y_i \geq \sum_j a_j$$

$$x_{ij}, y_i \in \{0,1\}$$

The y_i is redundant but makes the continuous relaxation tighter.

This is a modeling “trick,” part of the folklore of modeling.

The resulting model

$$\min \sum_{i=1}^n c_i y_i$$

$$\sum_j a_j x_{ij} \leq Q_i y_i, \text{ all } i$$

The y_i is redundant but makes the continuous relaxation tighter.

This is a modeling “trick,” part of the folklore of modeling.

$$0 \leq x_{ij} \leq y_i, \text{ all } i, j$$

$$\sum_{i=1}^n x_{ij} = 1, \text{ all } j$$

Conventional modeling wisdom would not use this constraint, because it is the sum of the first constraint over i .

But it radically reduces solution time, because it generates lifted knapsack cuts.

$$\sum_{i=1}^n Q_i y_i \geq \sum_j a_j$$

$$x_{ij}, y_i \in \{0,1\}$$

Research issues

- Can the **simplification** of a convex hull MILP formulation be automated?
- What are some conditions under which a **big-M** disjunctive model is a convex hull formulation?
- When does convex hull formulation of **logical constraints** tighten the model?
- How can a **modeling system** facilitate and encourage principled modeling?

Idea behind the convex hull formulation

Start by formulating a convex hull formulation of the relaxation of the disjunction...

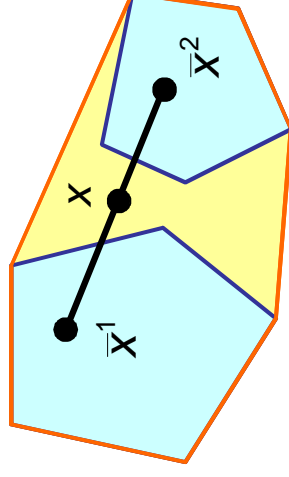
$$A^k \bar{x}^k \geq b^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k y_k \bar{x}^k$$

$$y_k \in [0,1]$$

Write each solution as a convex combination of points in the polyhedron



Convex hull relaxation

Idea behind the convex hull formulation

Now apply a change of variable

$$A^k x^k \geq b^k y_k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k x^k$$

$$y_k \in [0,1]$$

Change of variable

$$x = y_k \bar{x}^k$$

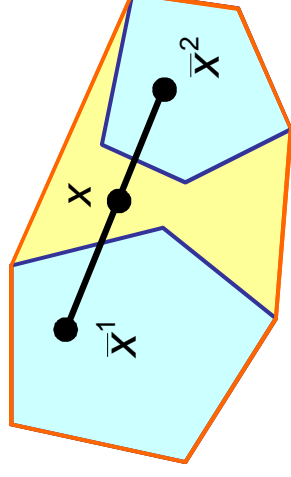
$$A^k \bar{x}^k \geq b^k, \text{ all } k$$

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$$x = \sum_k y_k \bar{x}^k$$

$$y_k \in [0,1]$$

Write each solution as a convex combination of points in the polyhedron



Convex hull relaxation

Idea behind the convex hull formulation

Now make y_k s 0-1 variables to get an MILP representation

$$A^k \bar{x}^k \geq b^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k y_k \bar{x}^k$$

$$y_k \in \{0,1\}$$

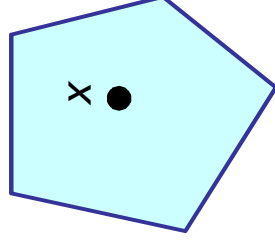
Make y_k s 0-1

$$A^k x^k \geq b^k y_k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k x^k$$

$$y_k \in [0,1]$$



Convex hull formulation

Idea behind the convex hull formulation

When is this a valid formulation?

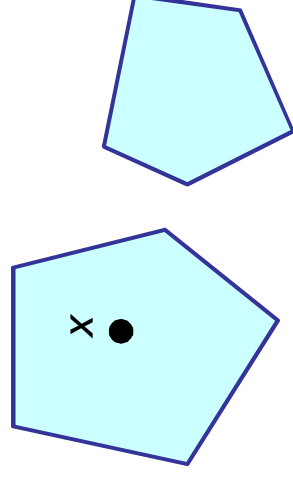
Let's look at an example first...

$$A^k \bar{x}^k \geq b^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k y_k \bar{x}^k$$

$$y_k \in \{0,1\}$$



Convex hull formulation

To simplify, replace x_1^2 with x_1
since $x_1^1 = 0$

$$\begin{aligned} \min \quad & x_2 \\ & x_1^1 = 0 \quad x_1^2 \geq 0 \\ & x_2^1 \geq 0 \quad -cx_1^2 + x_2^2 \geq fy_2 \\ & y_1 + y_2 = 1, \quad y_k \in [0,1] \\ & x_1 = x_1^1 + x_1^2, \quad x_2 = x_2^1 + x_2^2 \end{aligned}$$

To simplify, replace x_1^2 with x_1
since $x_1^1 = 0$

$$\begin{aligned} \min x_2 & & x_1 & \geq 0 \\ x_2^1 & \geq 0 & -cx_1 + x_2^2 & \geq fy_2 \\ y_1 + y_2 & = 1, & y_k & \in [0,1] \\ x_2 & = x_2^1 + x_2^2 \end{aligned}$$

Replace x_2^2 with x_2
because x_2^1 plays no role in the model

$$\begin{aligned} \min x_2 & & x_1 & \geq 0 \\ x_2^1 & \geq 0 & -cx_1 + x_2^2 & \geq fy_2 \\ y_1 + y_2 & = 1, & y_k & \in [0,1] \\ x_2 & = x_2^1 + x_2^2 \end{aligned}$$

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Replace y_2 with y
because y_1 plays no role in the model

$$\begin{aligned} \min \quad & x_2 \\ & x_1 \geq 0 \\ & -cx_1 + x_2 \geq fy_2 \\ & y_1 + y_2 = 1, \quad y_k \in [0,1] \end{aligned}$$

Replace y_2 with y
because y_1 plays no role in the model

$$\begin{aligned} \min \quad & x_2 \\ & x_1 \geq 0 \\ & x_2 \geq cx_1 + fy \\ & y \in [0,1] \end{aligned}$$

This simplifies as before...

$$\begin{aligned} \min \quad & x_2 \\ 0 \leq & x_1 \leq My \\ x_2 \geq & cx_1 + fy \\ y \in & \{0,1\} \end{aligned}$$

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Previous model

$$\begin{aligned} \min \quad & x_2 \\ x_1 \geq & 0 \\ x_2 \geq & cx_1 + fy \\ y \in & \{0,1\} \end{aligned}$$

This simplifies as before...

$$\begin{aligned} \min \quad & x_2 \\ & 0 \leq x_1 \leq My \\ & x_2 \geq cx_1 + fy \\ & y \in \{0,1\} \end{aligned}$$

or

$$\begin{aligned} \min \quad & cx + fy \\ & 0 \leq x \leq My \\ & y \in \{0,1\} \end{aligned}$$

“Big M”

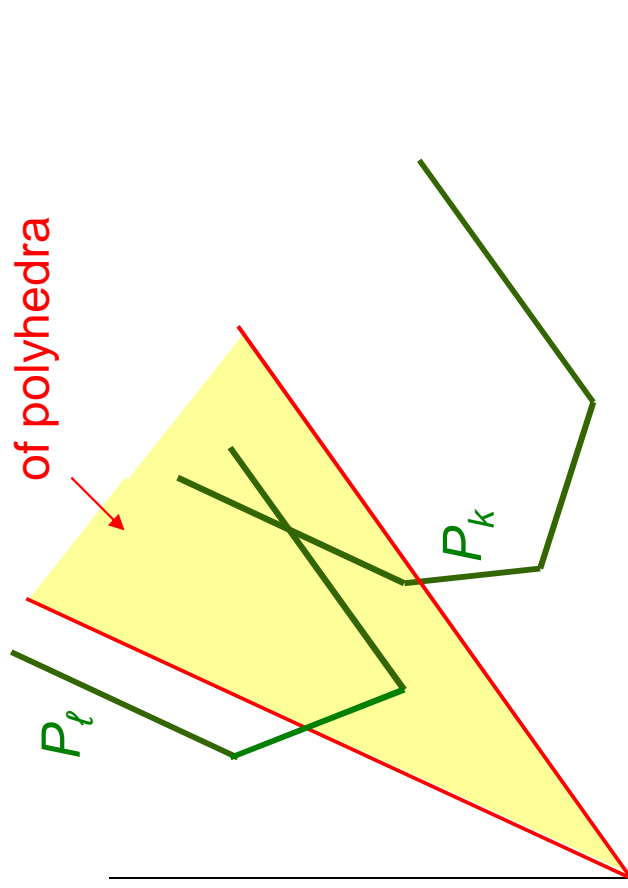
Previous model

$$\begin{aligned} \min \quad & x_2 \\ & x_1 \geq 0 \\ & x_2 \geq cx_1 + fy \\ & y \in \{0,1\} \end{aligned}$$

Why the disjunctive model works

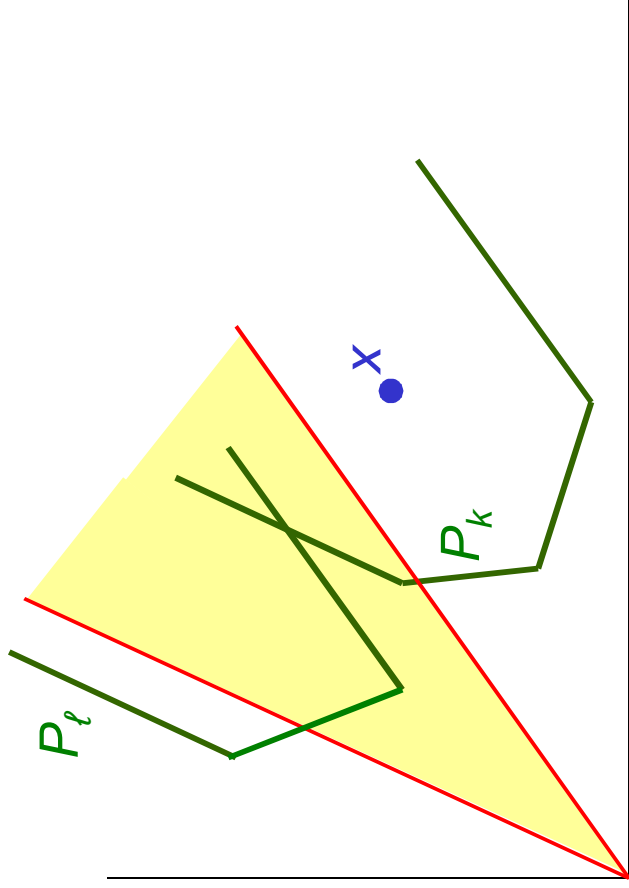
$$\begin{aligned} \min \quad & cx \\ & A^k x^k \geq b^k y_k, \text{ all } k \\ & \sum_k y_k = 1 \\ & x = \sum_k x^k \\ & y_k \in \{0,1\} \end{aligned}$$

Recession cone
of polyhedra



Let S be feasible set.

Why the disjunctive model works

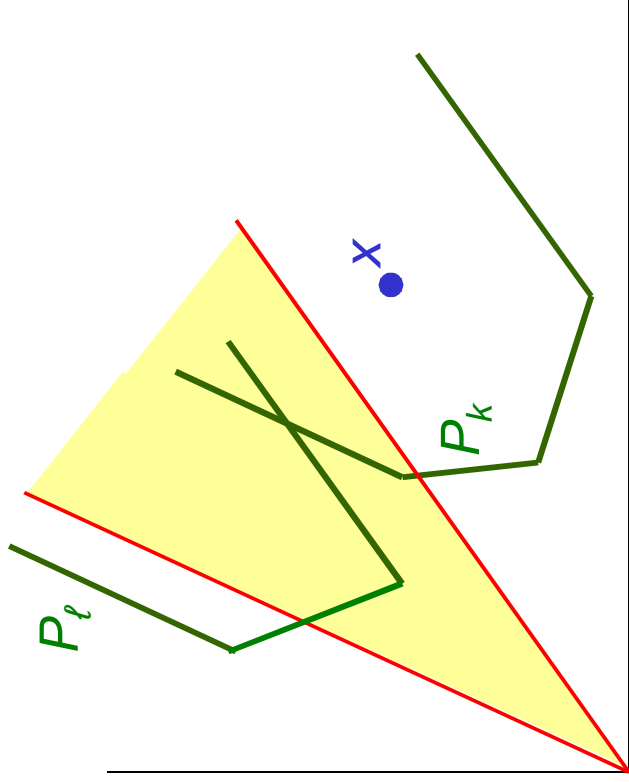


$$\begin{aligned} \min \quad & cx \\ & A^k x^k \geq b^k y_k, \text{ all } k \\ & \sum_k y_k = 1 \\ & x = \sum_k x^k \\ & y_k \in \{0,1\} \end{aligned}$$

Let S be feasible set.

$$x \in S \Rightarrow x \in \text{some } P_k$$

Why the disjunctive model works



$$\begin{aligned} \min \quad & cx \\ & A^k x^k \geq b^k y_k, \text{ all } k \\ & \sum_k y_k = 1 \\ & x = \sum_k x^k \\ & y_k \in \{0,1\} \end{aligned}$$

Let S be feasible set.

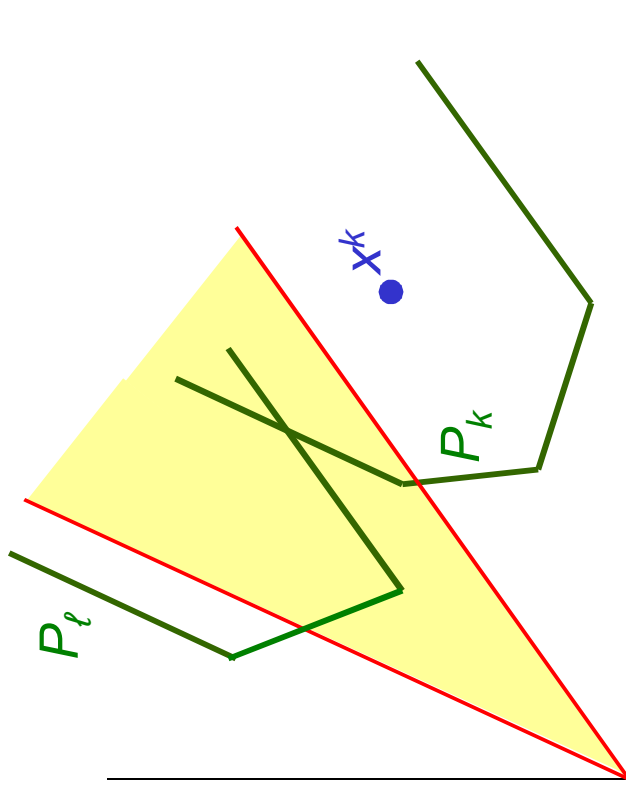
$$x \in S \Rightarrow x \in \text{some } P_k$$

\Rightarrow x satisfies the model for

$$y_k = 1, \text{ other } y_\ell = 0$$

$$x^k = x, \text{ other } x^\ell = 0$$

Why the disjunctive model works



$\min cx$

$A^k x^k \geq b^k y_k, \text{ all } k$

$\sum_k y_k = 1$

$x = \sum_k x^k$

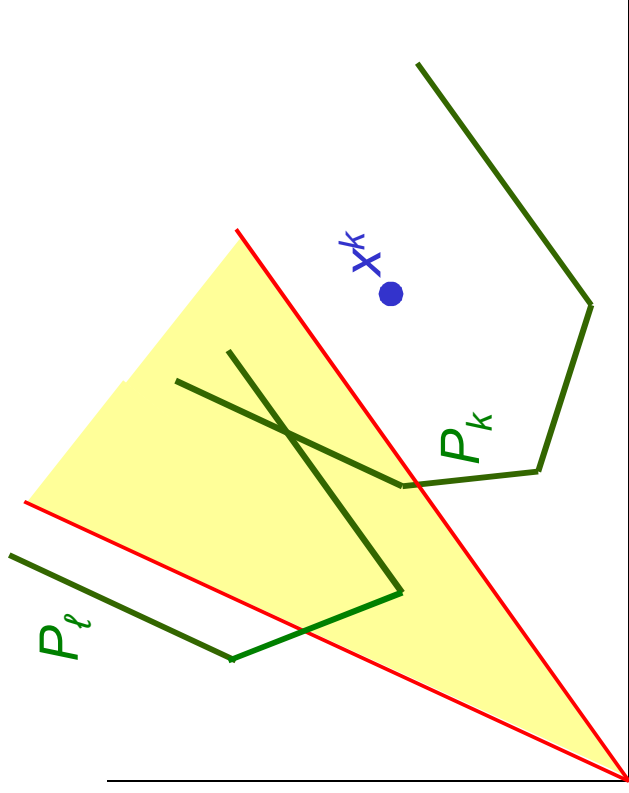
$y_k \in \{0,1\}$

Conversely, suppose

x, y, x^k s satisfy the model

\Rightarrow some $y_k = 1 \Rightarrow x^k \in P_k$

Why the disjunctive model works



min cx

$A^k x^k \geq b^k y_k$, all k

$\sum_k y_k = 1$

$x = \sum_k x^k$

$y_k \in \{0,1\}$

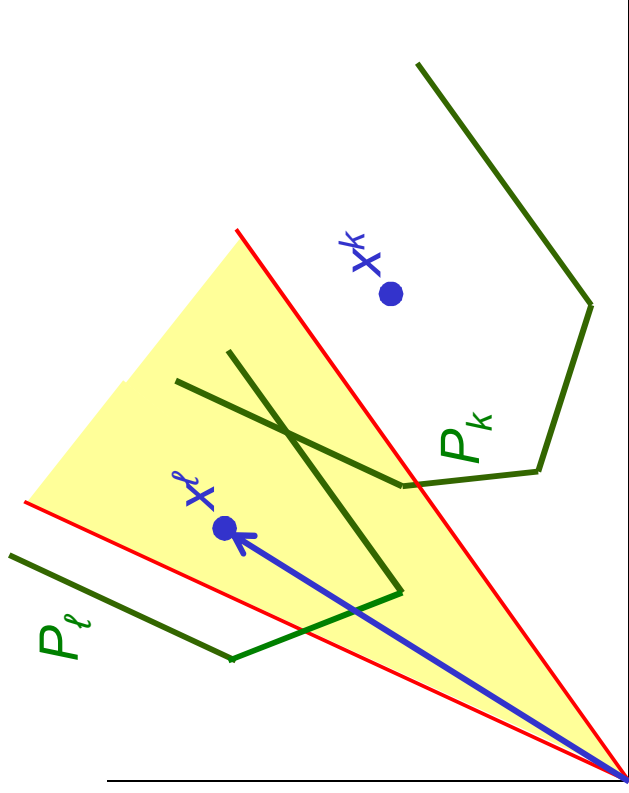
Conversely, suppose

x, y, x^k s satisfy the model

\Rightarrow some $y_k = 1 \Rightarrow x^k \in P_k$

$\Rightarrow A^\ell x^\ell \geq 0$ for other ℓ s

Why the disjunctive model works



$$\begin{aligned} \min \quad & cx \\ & A^k x^k \geq b^k y_k, \text{ all } k \\ & \sum_k y_k = 1 \\ & x = \sum_k x^k \\ & y_k \in \{0,1\} \end{aligned}$$

Conversely, suppose

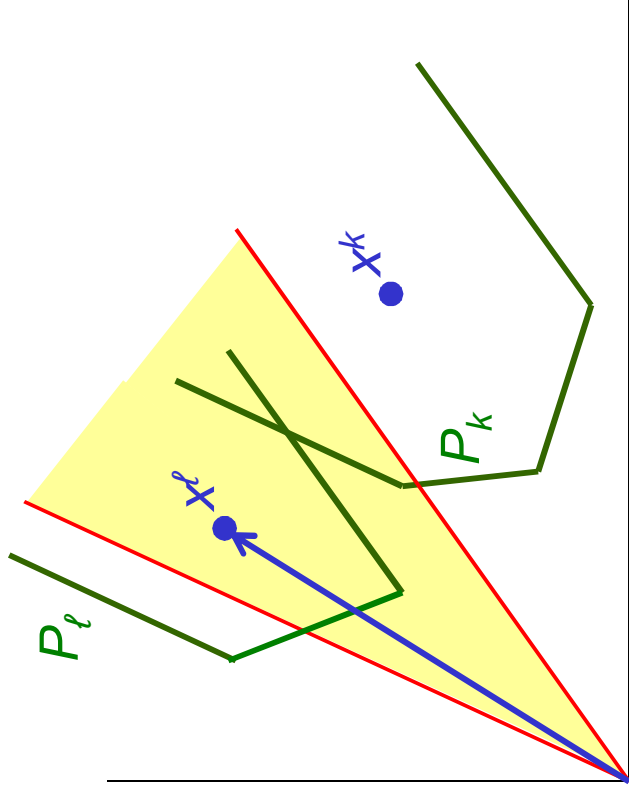
x, y, x^k 's satisfy the model

$$\Rightarrow \text{some } y_k = 1 \Rightarrow x^k \in P_k$$

$$\Rightarrow A^\ell x^\ell \geq 0 \text{ for other } \ell$$

$\Rightarrow x^\ell$'s are recession directions for other P_ℓ 's

Why the disjunctive model works



$$\begin{aligned} \min \quad & cx \\ & A^k x^k \geq b^k y_k, \text{ all } k \\ & \sum_k y_k = 1 \\ & x = \sum_k x^k \\ & y_k \in \{0,1\} \end{aligned}$$

Conversely, suppose

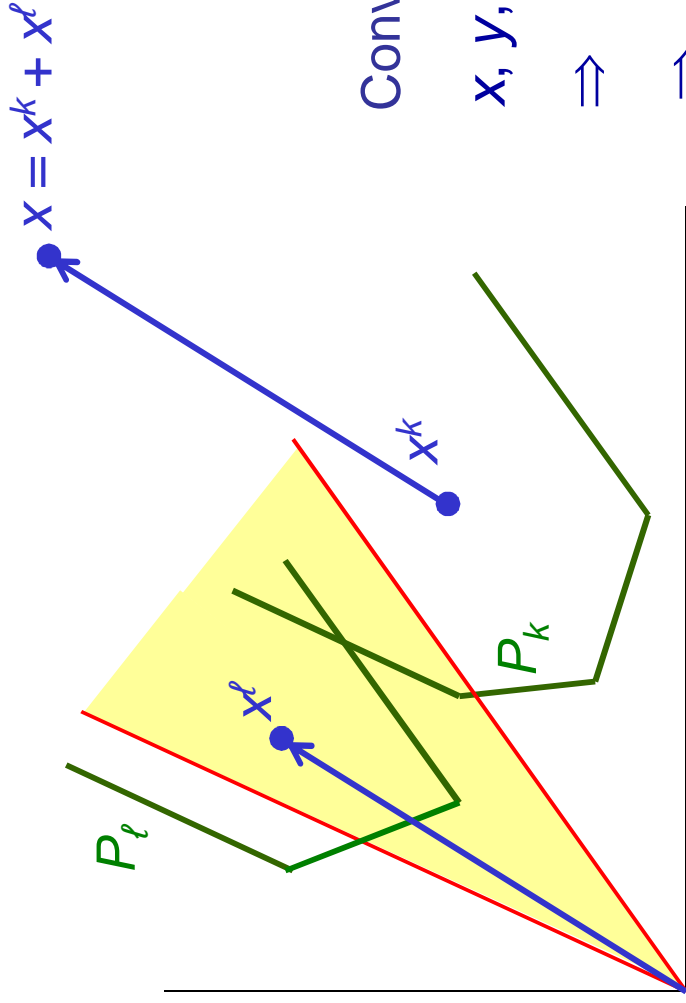
x, y, x^k 's satisfy the model

$$\Rightarrow \text{some } y_k = 1 \Rightarrow x^k \in P_k$$

$$\Rightarrow A^l x^l \geq 0 \text{ for other } l$$

$$\Rightarrow x^l \text{ 's are recession directions for } P_k$$

Why the disjunctive model works



$$\min cx$$

$$A^k x^k \geq b^k y_k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k x^k$$

$$y_k \in \{0,1\}$$

Conversely, suppose

x, y, x^k s satisfy the model

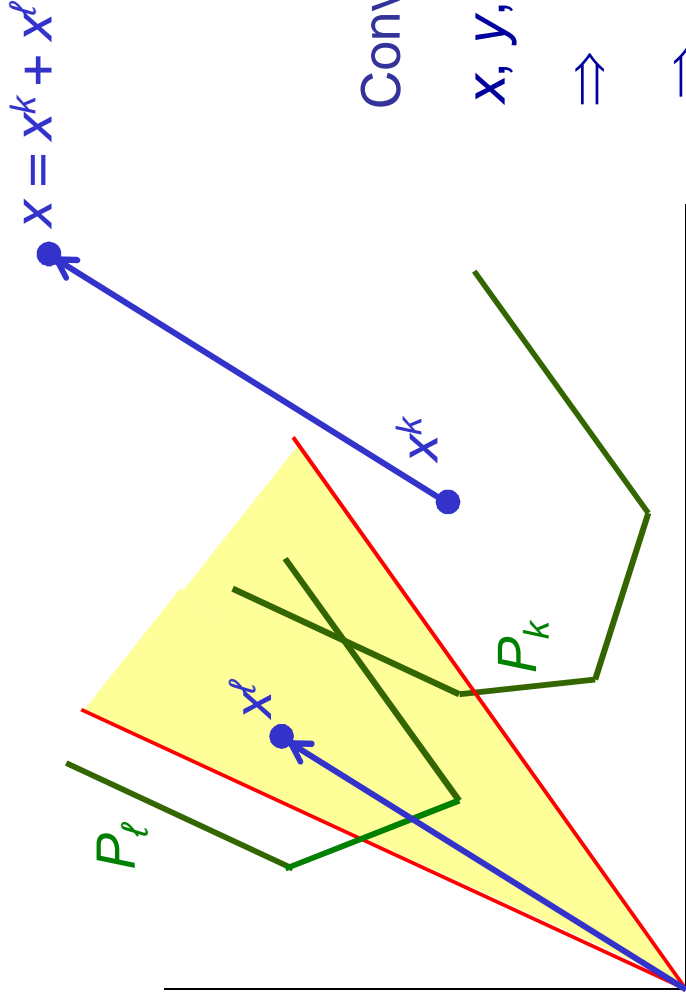
$$\Rightarrow \text{some } y_k = 1 \Rightarrow x^k \in P_k$$

$$\Rightarrow A^l x^l \geq 0 \text{ for other } l\text{s}$$

$$\Rightarrow x^l\text{s are recession directions for } P_k$$

$$\Rightarrow A^k x^l \geq 0 \Rightarrow A^k x = A^k \left(x^k + \sum_{l \neq k} x^l \right) \geq b^k$$

Why the disjunctive model works



$$\min cx$$

$$A^k x^k \geq b^k y_k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k x^k$$

$$y_k \in \{0,1\}$$

Conversely, suppose

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$$\Rightarrow A^k x^l \geq 0 \Rightarrow A^k x = A^k \left(x^k + \sum_{l \neq k} x^l \right) \geq b^k$$

$$\Rightarrow x \in P_k \Rightarrow x \in S$$

Multiple disjunctions

Combining individual convex hull formulations for two disjunctions...

$$\bigvee_k (A^k x \geq a^k)$$

$$\bigvee_k (B^k x \geq b^k)$$

does not necessarily produce a convex hull formulation for the pair...

Theorem. ...unless the disjunctions have no common variables.

To simplify, define

$$Z_t = Y_{t1}$$

$$Y_t = Y_{t2}$$

Convex hull MILP model of disjunction:

$$V_t^1 \geq f_t Y_{t1} \quad V_t^2 \geq 0 \quad V_t^3 \geq 0$$

$$0 \leq x_t^1 \leq C_t Y_{t1} \quad 0 \leq x_t^2 \leq C_t Y_{t2} \quad x_t^3 = 0$$

$$V_t = \sum_{k=1}^3 V_t^k, \quad x_t = \sum_{k=1}^3 x_t^k, \quad y_t = \sum_{k=1}^3 y_{tk}$$

$$y_{tk} \in \{0,1\}, \quad k = 1,2,3$$

To simplify, define

$$Z_t = Y_{t1}$$

$$Y_t = Y_{t2}$$

Convex hull MILP model of disjunction:

$$V_t^1 \geq f_t z_t \quad V_t^2 \geq 0 \quad V_t^3 \geq 0$$

$$0 \leq x_t^1 \leq C_t z_t \quad 0 \leq x_t^2 \leq C_t y_t \quad x_t^3 = 0$$

$$V_t = \sum_{k=1}^3 V_t^k, \quad x_t = \sum_{k=1}^3 x_t^k, \quad z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

= 1 for startup

= 1 for continued
production

$$\begin{array}{l} \text{Since } x_t^3 = 0 \\ \text{set } x_t = x_t^1 + x_t^2 \end{array}$$

Convex hull MILP model of disjunction:

$$\begin{array}{lll} v_t^1 \geq f_t z_t & v_t^2 \geq 0 & v_t^3 \geq 0 \\ 0 \leq x_t^1 \leq C_t z_t & 0 \leq x_t^2 \leq C_t y_t & x_t^3 = 0 \end{array}$$

$$v_t = \sum_{k=1}^3 v_t^k, \quad x_t = \sum_{k=1}^3 x_t^k, \quad z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

Since $x_t^3 = 0$
set $x_1 = x_1^1 + x_2^2$

Convex hull MILP model of disjunction:

$$v_t^1 \geq f_t z_t \quad v_t^2 \geq 0 \quad v_t^3 \geq 0$$

$$0 \leq x_t \leq C_t(z_t + y_t)$$

$$v_t = \sum_{k=1}^3 v_t^k, \quad z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

Since v_t occurs positively in the objective function, and v_t^2, v_t^3 do not play a role, let $v_t = v_t^1$

Convex hull MILP model of disjunction:

$$v_t^1 \geq f_t z_t \quad v_t^2 \geq 0 \quad v_t^3 \geq 0$$

$$0 \leq x_t \leq C_t(z_t + y_t)$$

$$v_t = \sum_{k=1}^3 v_t^k, \quad z_t + y_t \leq 1$$

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Since v_t occurs positively in the objective function, and V_t^2, V_t^3 do not play a role, let $V_t = V_t^1$

Convex hull MILP model of disjunction:

$$v_t \geq f_t z_t$$

$$0 \leq x_t \leq C_t(z_t + y_t)$$

$$z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

Logical variables



To tighten an MILP formulation of

$$A \vee B \vee C \vee D$$

$$E \vee F \vee G$$

$$(Y_A \wedge Y_B) \rightarrow Y_E$$

Put logical constraint in CNF:

$$\neg Y_A \vee \neg Y_B \vee Y_E$$

Replace negative with positive variables:

$$C \vee D \vee E$$

And add convex hull formulation of this clause.

Conjecture: this does not tighten the formulation when the disjunctions have no variables in common.

Why a Single Recession Cone

Suppose S is $Ax + Bz + Dy \geq b$ For each binary \bar{y} , this
represented by $x \in \mathbb{R}^n \times \mathbb{Z}^p$, $z \in \mathbb{R}^m$ describes a mixed integer
 $y \in \{0, 1\}^q$ polyhedron $P(\bar{y})$.

So S is a union of mixed integer polyhedra.

Why a single recession cone

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So S is a union of mixed integer polyhedra. Now x' is a recession
direction of nonempty $P(\bar{y})$ iff (x', u', y') is a recession direction of

$$\left\{ \begin{array}{l} x \\ u \\ y \end{array} \right\} \in \mathbb{R}^n \times \mathbb{Z}^p \times \mathbb{R}^{m+q} : \left[\begin{array}{ccc} A & B & D \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{array} \right] \begin{bmatrix} x \\ u \\ y \end{bmatrix} \geq \left[\begin{array}{c} 0 \\ 0 \\ \bar{y} \end{array} \right]$$

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That is, iff $\begin{bmatrix} A & B & D \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ u' \\ y' \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Why a single recession cone

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That is, iff $\begin{bmatrix} A & B & D \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ u' \\ y' \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ But this is independent of y .