Optimization Models for Fairness

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Modeling Fairness

• Why represent fairness in an optimization model?
  • In many applications, equitable distribution is an objective. How to formulate it mathematically?
  • Optimization models may provide insight into the consequences of ethical theories.
Modeling Equity

- Some applications
  - Single-payer health system.
  - Facility location (e.g., emergency services).
  - Taxation (revenue vs. progressivity).
  - Relief operations.
  - Telecommunications (lexmax, Nash bargaining solution)
Outline

• Optimization models and their implications
  • Utilitarian
  • Rawlsian (lexmax)
• Axiomatics
  • Deriving utilitarian and Rawlsian criteria
• Measures of inequality
• An allocation problem
• Bargaining solutions
  • Nash
  • Raiffa-Kalai-Smorodinsky
• Combining utility and equity
  • Health care example
Optimization Models and Their Implications

- Utilitarianism
  - The optimization problem
  - Characteristics of utilitarian allocations
  - Arguments for utilitarianism

- Rawlsian difference principle
  - The social contract argument
  - The lexmax principle
  - The optimization problem
  - Characteristics of lexmax solutions
Efficiency vs. Equity

- Two classical criteria for distributive justice:
  - Utilitarianism (efficiency)
  - Difference principle of John Rawls (equity)
- These have the must studied philosophical underpinnings.
Utilitarian Principle

• We assume that every individual has a utility function $v(x)$, where $x$ is the wealth allocation to the individual.
Utilitarian Principle

• A “just” distribution of wealth is one that maximizes total expected utility.

• Let $x_i =$ wealth initially allocated to person $i$
  
  $u_i(x_i) =$ utility eventually produced by person $i$
Utilitarian Model

- The utility maximization problem:

\[
\text{max } \sum_{i=1}^{n} u_i(x_i) \\
\sum_{i=1}^{n} x_i = 1 \\
x_i \geq 0, \text{ all } i
\]
Utilitarian Model

• Elementary analysis yields the optimal solution:

\[ u_1'(x_1) = \cdots = u_n'(x_n) \]

Marginal productivity

Distribute wealth so as to equalize marginal productivity.
Utilitarian Model

• Elementary analysis yields the optimal solution:

\[ u'_1(x_1) = \cdots = u'_n(x_n) \]

Distribute wealth so as to equalize marginal productivity.

• If we index persons in order of marginal productivity, i.e.,

\[ u'_i(\cdot) \leq u'_{i+1}(\cdot), \quad \text{all } i \]

Then less productive individuals receive less wealth.
Utilitarian Model

• Elementary analysis yields the optimal solution:

\[ u_1'(x_1) = \cdots = u_n'(x_n) \]

Marginal productivity

Distribute wealth so as to equalize marginal productivity.

• If we index persons in order of marginal productivity, i.e.,

\[ u_i'(\cdot) \leq u_{i+1}'(\cdot), \quad \text{all } i \]

Then less productive individuals receive less wealth.

• For convenience assume \( u_i(x_i) = c_i x_i^p \)
$p = 0.5$

Production functions $u_i(x_i)$ for 5 individuals
$p = 0.5$

Utility maximizing allocation
Utilitarian Model

• Classical utilitarian argument: concave utility functions tend to make the utilitarian solution more egalitarian.
Utilitarian Model

• Classical utilitarian argument: concave utility functions tend to make the utilitarian solution more egalitarian.

• A completely egalitarian allocation $x_1 = \cdots = x_n$ is optimal only when

$$u_1'(1/n) = \cdots = u_n'(1/n)$$

• So, equality is optimal only when everyone has the same marginal productivity in an egalitarian allocation.
Utilitarian Model

• Recall that \( u_i(x_i) = c_i x_i^p \) where \( p \geq 0 \)

• The optimal wealth allocation is

\[
x_i = c_i^{1-p} \left( \sum_{j=1}^{n} c_j^{1-p} \right)^{-1}
\]

• When \( p < 1 \):
  - Allocation is **completely egalitarian** only if \( c_1 = \cdots = c_n \)
  - Otherwise the **most egalitarian** allocation occurs when \( p \to 0 \): \( x_i = \frac{c_i}{\sum_j c_j} \)
Utilitarian Model

• The most egalitarian optimal allocation: people receive wealth in proportion to productivity $c_i$.
  
  − And this occurs only when productivity very insensitive to investment ($p \rightarrow 0$).

• Allocation can be very unequal when $p$ is closer to 1.
Utility maximizing wealth allocation

Wealth

Productivity coefficient $c_l$

$p = 0.9$
$p = 0.8$
$p = 0.7$
$p = 0.5$
$p = 0$
Utility Loss Due to Equality

Utility with equality/max utility vs. p
When output is proportional to investment, equality has high cost (cuts utility in half)
As $p \rightarrow 0$, optimal utility requires highly unequal allocation, but equal allocation is only slightly suboptimal.
Utilitarian Model

• More fundamentally, an egalitarian defense of utilitarianism is based on **contingency, not principle.**
  • If we **evaluate** the fairness of utilitarian distribution, then there must be **another standard** of equitable distribution.

• Utilitarianism can endorse:
  – Neglect of disabled or nonproductive people.
  – Meager wage for less talented people who work hard.
  – Fewer resources for people with less productive jobs. Not all jobs can be equally productive.

• if this results in greater total utility.
Rawlsian Difference Principle

- Rawls’ **Difference Principle** seeks to maximize the welfare of the worst off.
  - Also known as **maximin** principle.
  - Another formulation: inequality is permissible only to the extent that it is necessary to improve the welfare of those worst off.

\[
\max \min_i \{u_i(x_i)\} \\
\sum_i u_i(x_i) = 1 \\
x_i \geq 0, \text{ all } i
\]
Rawlsian Difference Principle

• The root idea is that when I make a decision for myself, I make a decision for anyone in similar circumstances.
  • It doesn’t matter who I am.

• Social contract argument
  • I make decisions (formulate a social contract) in an original position, behind a veil of ignorance as to who I am.
  • I must find the decision acceptable after I learn who I am.
  • I cannot rationally assent to a policy that puts me on the bottom, unless I would have been even worse off under alternative policies.
  • So the policy must maximize the welfare of the worst off.
Rawlsian Difference Principle

• Applies only to **basic goods**.
  • Things that people want, no matter what else they want.
  • Salaries, tax burden, medical benefits, etc.
  • For example, salary differentials may satisfy the principle if necessary to make the poorest better off.

• Applies to smallest **groups** for which outcome is predictable.
  • A lottery passes the test even though it doesn’t maximize welfare of worst off – the loser is unpredictable.
  • unless the lottery participants as a whole are worst off.
Rawlsian Difference Principle

• The difference rule implies a **lexmax** principle.
  – If applied recursively.

• **Lexmax (lexicographic maximum) principle:**
  – Maximize welfare of least advantaged class
  – then next-to-least advantaged class
  – and so forth.
Lexmax Model

• Applications
  • **Production planning** – Allocate scarce components to products to minimize worst-case delay to a customer.
  • **Location of fire stations** – Minimize worst-case response time.
  • **Workforce management** – Schedule rail crews so as to spread delays equitably over time. Similar for call center scheduling.
  • **Political districting** – Minimize worst-case deviation from proportional representation.
  • **Social planning** – Build a Rawlsian society.
Lexmax Model

• Assume each person’s share of total utility is proportional to the utility of his/her initial wealth allocation.
  – Thus individuals with more education, salary have greater access to social utility.

• Assume productivity functions $u_i(x_i) = c_i x_i^p$
  • Larger $p$ means productivity more sensitive to investment.

• Assume personal utility function $v(x_i) = x_i^q$
  • Larger $q$ means people care more about getting rich.
Lexmax Model

- The utility maximization problem:

\[
\begin{align*}
\text{lexmax } (y_1, \ldots, y_n) \\
y_i &= \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n \\
\sum_{i=1}^{n} y_i &= \sum_{i=1}^{n} u_i(x_i) \\
\sum_{i=1}^{n} x_i &= 1 \\
x_i &\geq 0, \quad \text{all } i
\end{align*}
\]
Lexmax Model

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\sum_{i=1}^{n} x_i = 1
\]

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x_i \geq 0, \quad \text{all } i
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Lexmax Model

- The utility maximization problem:

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\text{lexmax} \ (y_1, \ldots, y_n)
\]

\[
y_i = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n
\]

\[
\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i)
\]

\[
\sum_{i=1}^{n} x_i = 1 \quad \text{Budget}
\]

\[
x_i \geq 0, \quad \text{all } i
\]

Utility allocation to person \(i\)

Wealth allocation to person \(i\)
Lexmax Model

• The utility maximization problem:

\[
\begin{align*}
\text{lexmax } (y_1, \ldots, y_n) \\
y_i &= \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n \\
\sum_{i=1}^{n} y_i &= \sum_{i=1}^{n} u_i(x_i) \\
\sum_{i=1}^{n} x_i &= 1 \\
x_i &\geq 0, \text{ all } i
\end{align*}
\]

Utility allocation to person \(i\)

Wealth allocation to person \(i\)

\(y_i\)'s sum to total utility produced

Budget
Lexmax Model

- The utility maximization problem:

\[
\text{lexmax } (y_1, \ldots, y_n)
\]

\[
y_i = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n
\]

\[
\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i)
\]

\[
\sum_{i=1}^{n} x_i = 1
\]

\[
x_i \geq 0, \quad \text{all } i
\]
Lexmax Model

• The utility maximization problem:

\[
\text{lexmax } (y_1, \ldots, y_n) \\
y_i = \frac{v'(x_i)}{v'(x_1)}, \quad i = 2, \ldots, n \\
\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i) \\
\sum_{i=1}^{n} x_i = 1 \\
x_i \geq 0, \text{ all } i
\]

Theorem. If \( u_i'(\cdot) \leq u_{i+1}'(\cdot) \) and \( v(\cdot) \) is nondecreasing, this has an optimal solution in which \( y_1 \leq \cdots \leq y_n \).
Lexmax Model

• The utility maximization problem:

\[
\text{lexmax } (y_1, \ldots, y_n)
\]
\[
y_i = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n
\]
\[
\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i)
\]
\[
\sum_{i=1}^{n} x_i = 1
\]
\[
x_i \geq 0, \quad \text{all } i
\]

**Theorem.** If \( u_i'(\cdot) \leq u_{i+1}'(\cdot) \) and \( v(\cdot) \) is nondecreasing, this has an optimal solution in which \( y_1 \leq \cdots \leq y_n \)

Model now simplifies.
\( \rho = 0.5 \)  
Utility maximizing allocation 
\[ \beta = 0 \]
$p = q = 0.5$

Lexmax allocation

Production vs. Wealth allocation graph with different curves for various utilities.
$\rho = 0.5$

Utility Maximizing Allocation

$P = 0.5$
$p = q = 0.5$

Lexmax Allocation

Production vs. Wealth allocation graph with curves for $u_1$, $u_2$, $u_3$, $u_4$, and $u_5$. The graph shows how production changes with different wealth allocations.
Lexmax Model

• When does the Rawlsian model result in equality?
  – That is, when do we have $x_1 = \cdots = x_n$ in the solution of the lexmax problem?
Lexmax Model

- Conditions for equality at optimality:

\[ 2\mu_1 - \mu_2 = d_1 \]
\[ \mu_1 + \mu_i - \mu_{i+1} = d_i, \quad i = 2, \ldots, n-2 \]
\[ \mu_1 + \mu_{n-1} = d_{n-1} \]

- with RHS’s:

\[ d_i = v(x_i) \frac{\sum c_i u_i(x_i)}{\sum v(x_i)} \left( \frac{v'(x_i)}{v(x_i)} - \frac{u_{i+1}'(x_{i+1}) - u_1'(x_1)}{\sum c_i u_i(x_i)} + \frac{v'(x_{i+1}) - v'(x_1)}{\sum v(x_i)} \right) \]

- Remarkably, these can be solved in closed form, yielding
Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

\[
\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \leq \frac{q}{p} \cdot \frac{n-k}{k} \sum_{i=1}^{n} c_i
\]

for \( k = 1, \ldots, n - 1 \).
Lexmax Model

• **Theorem.** The lexmax distribution is egalitarian only if

\[
\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \leq \frac{q}{p} \cdot \frac{n-k}{k} \sum_{i=1}^{n} c_i
\]

for \(k = 1, \ldots, n-1\).

Average of \(n-k\) largest \(c_i\)'s

Average of \(k\) smallest \(c_i\)'s
Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

\[
\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \leq q \cdot \frac{n-k}{p} \frac{1}{k} \sum_{i=1}^{n} c_i
\]

for \( k = 1, \ldots, n - 1 \).

- **Equality is more likely** to be required when \( p \) is small.
  - When investment in an individual yields rapidly decreasing marginal returns.
Lexmax Model

• **Theorem.** The lexmax distribution is egalitarian only if

\[
\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \leq \frac{q}{p} \cdot \frac{n-k}{k} \sum_{i=1}^{n} c_i
\]

for \( k = 1, \ldots, n - 1 \).

• **Equality test is more sensitive** at upper end (large \( k \)).
  - Equality is **unlikely** to be required when there is a long upper tail (individuals at the top are very productive).
  - Equality **may be required** even when there is a long lower tail (individuals at the bottom are very unproductive).
Lexmax Model

• **Theorem.** The lexmax distribution is egalitarian only if
\[
\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \leq \frac{q}{p} \cdot \frac{n-k}{k} \sum_{i=1}^{n} c_i
\]
for \( k = 1, \ldots, n - 1 \).

• Equality is **more likely** to be required when \( q \) is large.
  – That is, when greater wealth yields rapidly increasing marginal utility.
  – That is, when people **want to get rich**.
Axiomatics

- Social welfare functions
- Interpersonal comparability
- Deriving the utilitarian criterion
- Deriving the maximin/minimax criterion
Axiomatics

• The economics literature derives social welfare functions from axioms of rational choice.
  • Some axioms are strong and hard to justify.
  • The social welfare function depends on degree of interpersonal comparability of utilities.
  • Arrow’s impossibility theorem was the first result, but there are many others.

• Social welfare function
  • A function $f(u_1, \ldots, u_n)$ of individual utilities.
  • Objective is to maximize $f(u_1, \ldots, u_n)$. 
Axiomatics

• Social Preferences
  • Let \( u = (u_1, \ldots, u_n) \) be the vector of utilities allocated to individuals.
  • A social welfare function ranks distributions: \( u \) is preferable to \( u' \) if \( f(u) > f(u') \).
Interpersonal Comparability

• Unit comparability
  • Suppose each individual’s utility $u_i$ is changed to $\beta u_i + \alpha_i$.
  • This doesn’t change the utilitarian ranking:

  $\sum_i u_i(x) > \sum_i u_i(y)$ if and only if

  $\sum_i (\beta u_i(x) + \alpha_i) > \sum_i (\beta u_i(y) + \alpha_i)$

• This is unit comparability.
• That is, changing units of measure and giving everyone a different zero point has no effect on ranking.
Interpersonal Comparability

• Unit comparability
  • Unit comparability is enough to make utilitarian calculations meaningful.
  • Given certain axioms, along with unit comparability, a utilitarian social welfare function is necessary
Axioms

• Anonymity
  • Social preferences are the same if indices of $u_i$ are permuted.

• Strict pareto
  • If $u > u'$, then $u$ is preferred to $u'$.

• Independence of irrelevant alternatives
  • The preference of $u$ over $u'$ depends only on $u$ and $u'$ and not on what other utility vectors are possible.

• Separability of unconcerned individuals
  • Individuals $i$ for which $u_i = u_i'$ don’t affect the ranking of $u$ and $u'$.
Axiomatics

Theorem
Given **unit comparability**, any social welfare function $f$ that satisfies the axioms has the form $f(u) = \Sigma_i a_i u_i$ (**utilitarian**).
Interpersonal Comparability

• Level comparability
  • Suppose each individual’s utility $u_i$ is changed to $\phi(u_i)$, where $\phi$ is a monotone increasing function.
  • This doesn’t change the maximin ranking:

  $$\min_i \{u(x_i)\} > \min_i \{u(y_i)\} \text{ if and only if } \min_i \{\phi(u(x_i))\} > \min_i \{\phi(u(y_i))\}$$

• This is level comparability.
Axiomatics

• Level comparability
  • Level comparability is enough to make maximin comparisons meaningful.

Theorem
Given level comparability, any social welfare function that satisfies the axioms leads to a maximin or minimax criterion.
Axiomatics

• Problem with utilitarian theorem
  • The assumption of unit comparability implies no more than unit comparability.
  • This is almost the same as assuming utilitarianism.
  • It rules out a maximin criterion from the start, because the “worst-off” is a meaningless concept.

• Problem with maximin theorem
  • The assumption of level comparability implies no more than level comparability.
  • This rules out utilitarianism from the start.
Measures of Inequality

• An example
  • Utilitarian, maximin, and lexmax solution

• Inequality measures
  • Relative range, max, min
  • Relative mean deviation
  • Variance, coefficient of variation
  • McLoone index
  • Gini coefficient
  • Atkinson index
  • Hoover index
  • Theil index
Measures of Inequality

• Assume we wish to minimize inequality.
  • We will survey several measures of inequality.
  • They have different strengths and weaknesses.
  • Minimizing inequality may result in less total utility.

• Pigou-Dalton condition.
  • One criterion for evaluating an inequality measure.
  • If utility is transferred from one who is worse off to one who is better off, inequality should increase.
Measures of Inequality

• Applications
  • Tax policy
  • Disaster recovery
  • Educational funding
  • Greenhouse gas mitigation
  • Ramp metering on freeways
Example

Production functions for 5 individuals

Utilities vs. Resources
Utilitarian

$$\max \sum_{i} u_i$$

LP model:  $$\max \sum_{i=1}^{5} u_i$$
$$u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \text{all } i, \ \sum_i x_i = B$$

where  $$(a_1,\ldots,a_5) = (0.5, 0.75, 1, 1.5, 2)$$
$$(b_1,\ldots,b_5) = (20, 25, 30, 35, 40)$$
$$B = 100$$
Utilitarian
Rawlsian

\[ \max \left\{ \min_i \{u_i\} \right\} \]

LP model: \[ \max u_{\min} + \epsilon \sum_i u_i \]
\[ u_{\min} \leq u_i, \text{ all } i \]
\[ u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \text{ all } i, \sum_i x_i = B \]

Ensures that solution is Pareto optimal
Rawlsian
Utilitarian
**Lexmax**

\[ \text{lexmax}\{u_1, \ldots, u_n\} \]

Sequence of LP models, 
\(k = 1, \ldots, n - 1:\)

\[
\begin{align*}
\max & \quad u_{\min} \\
\text{subject to} & \quad u_i = u_i^* \quad \text{all } i < k \\
& \quad u_{\min} \leq u_i \quad \text{all } i \geq k \\
& \quad u_i = a_i x_i \quad 0 \leq x_i \leq b_i \quad \text{all } i, \quad \sum_i x_i = B
\end{align*}
\]

Re-index for each \(k\) so that \(u_i\) for \(i < k\) were fixed in previous iterations.
Lexmax
Rawlsian

![Graph showing cumulative utility, utility, and resources over time. The graph has a y-axis ranging from 0 to 160 and an x-axis ranging from 1 to 5. There are three lines: blue for cumulative utility, red for utility, and green for resources. The cumulative utility line shows a steep increase after time point 4, while the utility and resources lines are relatively flat until time point 4, after which they also show an increase.]
Utilitarian
Relative Range

\[ \frac{U_{\text{max}} - U_{\text{min}}}{\bar{U}} \]

where \( u_{\text{max}} = \max_i \{u_i\} \quad u_{\text{min}} = \min_i \{u_i\} \quad \bar{u} = (1 / n) \sum_i u_i \)

Rationale:
• Perceived inequality is relative to the best off.
• A distribution should be judged by the position of the worst-off.
• Therefore, minimize gap between top and bottom.

Problems:
• Ignores distribution between extremes.
• Violates Pigou-Dalton condition
Relative Range

$$\frac{U_{\text{max}} - U_{\text{min}}}{\bar{U}}$$

This is a fractional linear programming problem.

Use Charnes-Cooper transformation to an LP. In general,

$$\min \frac{cx + c_o}{dx + d_o}$$

becomes

$$\min cx' + c_0z$$

$$Ax \geq b$$

$$d'x' + d_0z = 1$$

$$x', z \geq 0$$

after change of variable $x = x'/z$ and fixing denominator to 1.
Relative Range

\[ \frac{U_{\max} - U_{\min}}{\bar{U}} \]

Fractional LP model:

\[
\min \frac{u_{\max} - u_{\min}}{(1/ n)\sum_i u_i} \\
\]

\[ u_{\max} \geq u_i, \; u_{\min} \leq u_i, \; \text{all } i \]

\[ u_i = a_i x_i, \; 0 \leq x_i \leq b_i, \; \text{all } i, \; \sum_i x_i = B \]

LP model:

\[
\min u_{\max} - u_{\min} \\
\]

\[ u_{\max} \geq u_i, \; u_{\min} \leq u_i, \; \text{all } i \]

\[ u_i' = a_i x_i', \; 0 \leq x_i' \leq b_i z, \; \text{all } i, \; \sum_i x_i' = Bz \]

\[ (1/ n)\sum_i u_i' = 1 \]
Relative Range
Lexmax
Relative Max

\[ \frac{u_{\text{max}}}{\bar{u}} \]

**Rationale:**
- Perceived inequality is relative to the best off.
- Possible application to salary levels (typical vs. CEO)

**Problems:**
- Ignores distribution below the top.
- Violates Pigou-Dalton condition
Relative Max

\[ \frac{U_{\text{max}}}{U} \]

Fractional LP model:

\[
\min \frac{u_{\text{max}}}{(1/ n) \sum_i u_i} \\
u_{\text{max}} \geq u_i, \text{ all } i \\
u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \text{ all } i, \ \sum_i x_i = B
\]

LP model:

\[
\min u_{\text{max}} \\
u_{\text{max}} \geq u'_i, \text{ all } i \\
u'_i = a_i x'_i, \ 0 \leq x'_i \leq b_i z, \text{ all } i, \ \sum_i x'_i = B z \\
(1/ n) \sum_i u'_i = 1
\]
Relative Max

![Graph showing cumulative utility, utility, and resources over time]
Relative Range
Relative Min

\[ \frac{u_{\text{min}}}{\bar{u}} \]

**Rationale:**
- Measures adherence to Rawlsian Difference Principle.
- relativized to mean

**Problems:**
- Ignores distribution above the bottom.
- Violates Pigou-Dalton condition
Relative Min

\[ \frac{U_{\text{min}}}{U} \]

Fractional LP model:

\[ \max \frac{u_{\text{min}}}{(1/ n)} \sum_i u_i \]

\[ u_{\text{min}} \leq u_i, \text{ all } i \]

\[ u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \text{ all } i, \ \sum_i x_i = B \]

LP model:

\[ \max u_{\text{min}} \]

\[ u_{\text{min}} \geq u_i', \text{ all } i \]

\[ u_i' = a_i x_i', \ 0 \leq x_i' \leq b_iz, \text{ all } i, \ \sum_i x_i' = Bz \]

\[ (1/ n) \sum_i u_i' = 1 \]
Relative Min
Relative Max
Relative Range
Relative Mean Deviation

\[
\frac{\sum_{i} |u_i - \bar{u}|}{\bar{u}}
\]

Rationale:
- Perceived inequality is relative to average.
- Entire distribution should be measured.

Problems:
- Violates Pigou-Dalton condition
- Insensitive to transfers on the same side of the mean.
- Insensitive to placement of transfers from one side of the mean to the other.
Relative Mean Deviation

\[
\sum_{i} |u_i - \bar{u}| \over \bar{u}
\]

Fractional LP model:

\[
\max \frac{\sum (u_i^+ + u_i^-)}{\bar{u}}
\]

\[u_i^+ \geq u_i - \bar{u}, \ u_i^- \geq \bar{u} - u_i, \ \text{all } i\]

\[\bar{u} = (1/ n) \sum_{i} u_i\]

\[u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum_{i} x_i = B\]

LP model:

\[
\max \sum_{i} (u_i^+ + u_i^-)
\]

\[u_i^+ \geq u_i - 1, \ u_i^- \leq u_i - 1, \ \text{all } i\]

\[(1/ n) \sum_{i} u_i' = 1\]

\[u_i' = a_i x_i', \ 0 \leq x_i' \leq b_i z, \ \text{all } i, \ \sum_{i} x_i' = B z\]
Relative Mean Deviation
Relative Range
Variance

\[
\frac{1}{n} \sum_{i} (u_i - \bar{u})^2
\]

**Rationale:**

- Weight each utility by its distance from the mean.
- Satisfies Pigou-Dalton condition.
- Sensitive to transfers on one side of the mean.
- Sensitive to placement of transfers from one side of the mean to the other.

**Problems:**

- Weighting is arbitrary?
- Variance depends on scaling of utility.
Variance

\[ \frac{1}{n} \sum_{i} (u_i - \bar{u})^2 \]

Convex nonlinear model:

\[ \min \left( \frac{1}{n} \sum_{i} (u_i - \bar{u})^2 \right) \]

\[ \bar{u} = \frac{1}{n} \sum_{i} u_i \]

\[ u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all} \ i, \ \sum_{i} x_i = B \]
Variance
Relative Mean Deviation
Relative Range
Coefficient of Variation

\[
\left( \frac{1}{n} \sum_{i} (u_i - \bar{u})^2 \right)^{1/2} \frac{1}{\bar{u}}
\]

**Rationale:**
- Similar to variance.
- Invariant with respect to scaling of utilities.

**Problems:**
- When minimizing inequality, there is an incentive to reduce average utility.
- Should be minimized only for fixed total utility.
Coefficient of Variation

\[
\left( \frac{1}{n} \sum_i (u_i - \bar{u})^2 \right)^{1/2}
\]

\[\frac{1}{\bar{u}}\]

Again use change of variable \( u = u'/z \) and fix denominator to 1.

\[
\min \left( \frac{(1/ n) \sum_i (u_i - \bar{u})^2}{\bar{u}} \right)^{1/2}
\]

becomes

\[
\min \left( \frac{(1/ n) \sum_i (u'_i - 1)^2}{1} \right)^{1/2}
\]

\( Au' \geq bz \)

\( (1/ n) \sum_i u'_i = 1 \)

\( u' \geq 0 \)

Can drop exponent to make problem convex
Coefficient of Variation

\[
\left( \frac{(1 / n) \sum_i (u_i - \bar{u})^2}{\bar{u}} \right)^{1/2}
\]

Fractional nonlinear model:

\[
\max \left( \frac{(1 / n) \sum_i (u_i - \bar{u})^2}{\bar{u}} \right)^{1/2}
\]

\[\bar{u} = (1 / n) \sum_i u_i\]

\[u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum_i x_i = B\]

Convex nonlinear model:

\[
\min (1 / n) \sum_i (u'_i - 1)^2
\]

\[\frac{1}{n} \sum_i u'_i = 1\]

\[u'_i = a_i x'_i, \ 0 \leq x'_i \leq b_i z, \ \text{all } i, \ \sum_i x'_i = Bz\]
Coefficient of Variation
Variance
Relative Mean Deviation
McLoone Index

\[
\frac{(1/2) \sum_{i: u_i < m} u_i}{\bar{u}}
\]

**Rationale:**
- Ratio of average utility below median to overall average.
- No one wants to be “below average.”
- Pushes average up while pushing inequality down.

**Problems:**
- Violates Pigou-Dalton condition.
- Insensitive to upper half.
McLoone Index

\[
(1/2) \sum_{i: u_i < m} u_i \\
\overline{U}
\]

Fractional MILP model:

\[
\max \frac{\sum v_i}{\sum u_i}
\]

Defines median \( m \)

Defines \( v_i = u_i \) if \( u_i \) is below median

Half of utilities are below median

Selects utilities below median

\[
m - My_i \leq u_i \leq m + M(1 - y_i), \text{ all } i
\]

\[
v_i \leq u_i, v_i \leq My_i, \text{ all } i
\]

\[
\sum_i y_i < n/2
\]

\[
u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \text{ all } i, \quad \sum_i x_i = B
\]

\[
y_i \in \{0,1\}, \text{ all } i
\]
McLoone Index

\[
\frac{(1/2) \sum_{i: u_i < m} u_i}{\bar{u}}
\]

MILP model:

\[
\begin{align*}
\text{max} & \quad \sum_i v'_i \\
 m' - M y_i & \leq u'_i \leq m' + M(1 - y_i), \quad \text{all } i \\
v'_i & \leq u'_i, \quad v'_i \leq M y_i, \quad \text{all } i \\
\sum_i y_i & < n / 2 \\
u'_i & = a_i x'_i, \quad 0 \leq x'_i \leq b_i z, \quad \text{all } i, \quad \sum_i x'_i = B z \\
y_i & \in \{0, 1\}, \quad \text{all } i
\end{align*}
\]
McLoone Index
Relative Min

![Relative Min Chart]
Gini Coefficient

\[
(1/ n^2) \sum_{i,j} \left| u_i - u_j \right| \frac{1}{2 \bar{u}}
\]

Rationale:

- Relative mean difference between all pairs.
- Takes all differences into account.
- Related to area above cumulative distribution (Lorenz curve).
- Satisfies Pigou-Dalton condition.

Problems:

- Insensitive to shape of Lorenz curve, for a given area.
Gini Coefficient

\[
\frac{(1/ n^2) \sum_{i,j} |u_i - u_j|}{2\bar{u}}
\]

Cumulative utility

Gini coeff. = \( \frac{\text{blue area}}{\text{area of triangle}} \)

Lorenz curve
Gini Coefficient

\[ \frac{(1/ n^2) \sum_{i,j} |u_i - u_j|}{2\bar{u}} \]

Fractional LP model:

\[ \max \frac{(1/ 2n^2) \sum_{ij} (u_{ij}^+ + u_{ij}^-)}{\bar{u}} \]

\[ u_{ij}^+ \geq u_i - u_j, \; u_{ij}^- \geq u_j - u_i, \; \text{all } i, j \]

\[ \bar{u} = (1/ n) \sum_i u_i \]

\[ u_i = a_i x_i, \; 0 \leq x_i \leq b_i, \; \text{all } i, \quad \sum_i x_i = B \]

LP model:

\[ \max (1/ 2n^2) \sum_{ij} (u_{ij}^+ + u_{ij}^-) \]

\[ u_{ij}^+ \geq u_i' - u_j', \; u_{ij}^- \geq u_j' - u_i', \; \text{all } i, j \]

\[ (1/ n) \sum_i u_i' = 1 \]

\[ u_i' = a_i x_i', \; 0 \leq x_i' \leq b_i z, \; \text{all } i, \quad \sum_i x_i' = Bz \]
Gini Coefficient
Coefficient of Variation
Variance
Gini Coefficient by Country (2013)
Historical Gini Coefficient, 1945-2010

Gini Index: Income Disparity since World War II

Note: Gini coefficients range from 0 (perfect equality) to 100 (perfect inequality).
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum_{i} \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

**Rationale:**
- Best seen as measuring inequality of resources \( x_i \).
- Assumes allotment \( y \) of resources results in utility \( y^p \).
- This is average utility per individual.
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum_{i} \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

**Rationale:**

- Best seen as measuring inequality of resources \( x_i \).
- Assumes allotment \( y \) of resources results in utility \( y^p \).
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.
Atkinson Index

$$1 - \left( \frac{1}{n} \sum_{i} \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p}$$

**Rationale:**

- Best seen as measuring inequality of resources $x_i$.
- Assumes allotment $y$ of resources results in utility $y^p$.
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.
- This is additional resources per individual necessary to sustain inequality.
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum_i \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

**Rationale:**
- \( p \) indicates “importance” of equality.
- Similar to \( L_p \) norm
- \( p = 1 \) means inequality has no importance
- \( p = 0 \) is Rawlsian (measures utility of worst-off individual).

**Problems:**
- Measures utility, not equality.
- Doesn’t evaluate distribution of utility, only of resources.
- \( p \) describes utility curve, not importance of equality.
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum_i \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

To minimize index, solve fractional problem

\[ \max \sum_i \left( \frac{x_i}{\bar{x}} \right)^p = \frac{\sum x_i^p}{\bar{x}^p} \]
\[ Ax \geq b, \ x \geq 0 \]

After change of variable \( x_i = x'_i / z \), this becomes

\[ \max \sum x'_i^p \]
\[ (1/n) \sum x'_i = 1 \]
\[ Ax' \geq bz, \ x' \geq 0 \]
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

Fractional nonlinear model:
\[ \max \frac{\sum x_i^p}{\bar{x}^p} \]
\[ \bar{x} = (1/n) \sum x_i \]
\[ \sum x_i = B, \quad x \geq 0 \]

Concave nonlinear model:
\[ \max \sum x_i'^p \]
\[ (1/n) \sum x_i' = 1 \]
\[ \sum x_i' = Bz, \quad x' \geq 0 \]
Atkinson index

![Diagram showing the Atkinson index with three lines: Cumulative Utility, Utility, and Resources. The x-axis represents the values 1 to 5, and the y-axis represents values from 0 to 160. The Cumulative Utility line is the most往上, followed by the Utility line, and the Resources line is the least往上.](image-url)
Hoover Index

\[
\sum_{i} \left| u_i - \bar{u} \right| \frac{1}{2} \left( \frac{1}{\sum_{i} u_i} \right)
\]

**Rationale:**
- Fraction of total utility that must be redistributed to achieve total equality.
- Proportional to maximum vertical distance between Lorenz curve and 45° line.
- Originated in regional studies, population distribution, etc. (1930s).
- Easy to calculate.

**Problems:**
- Less informative than Gini coefficient?
Cumulative utility =

\[ \sum_{i} \left( \frac{1}{2} \right) \sum_{i} \left| u_i - \bar{u} \right| \]

(1/2) \[
\frac{\sum_{i} u_i}{\sum_{i} u_i}
\]

Hoover Index

Hoover index = max vertical distance

Total utility = 1

Lorenz curve

Individuals ordered by increasing utility

Cumulative utility
Hoover Index
Gini Coefficient

![Graph showing Gini Coefficient]

Legend:
- **Cum. Utility**
- **Utility**
- **Resources**
Theil Index

\[(1 / n) \sum \left( \frac{u_i}{u} \ln \frac{u_i}{u} \right)\]

**Rationale:**
- One of a family of entropy measures of inequality.
- Index is zero for complete equality (maximum entropy).
- Measures nonrandomness of distribution.
- Described as stochastic version of Hoover index.

**Problems:**
- Motivation unclear.
- A. Sen doesn’t like it.
Theil Index

\[(1/ n) \sum_i \left( \frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)\]

Nasty nonconvex model:

\[
\min \left( \frac{1}{n} \right) \sum_i \left( \frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)
\]

\[\bar{u} = \left( \frac{1}{n} \right) \sum_i u_i\]

\[u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum_i x_i = B\]
Theil Index
Hoover Index
Gini Coefficient
An Allocation Problem

- From Yaari and Bar-Hillel, 1983.
- 12 grapefruit and 12 avocados are to be divided between Jones and Smith.
- How to divide justly?

Utility provided by one fruit of each kind

<table>
<thead>
<tr>
<th></th>
<th>Jones</th>
<th>Smith</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grapefruit</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>Avocado</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>
An Allocation Problem

The optimization problem:

\[
\begin{align*}
\text{max} \quad & f(u_1, u_2) \\
u_1 &= 100x_{11}, \quad u_2 = 50x_{12} + 50x_{22} \\
x_{i1} + x_{i2} &= 12, \quad i = 1, 2 \\
x_{ij} &\geq 0, \quad \text{all } i, j
\end{align*}
\]

where \( u_i \) = utility for person \( i \) (Jones, Smith) \\
\( x_{ij} \) = allocation of fruit \( i \) (grapefruit, avocados) \\
to person \( j \)
Utilitarian Solution

\[ f(u_1, u_2) = u_1 + u_2 \]

Smith’s utility

Jones’ utility

Optimal solution

(1200, 600)
Rawlsian (maximin) solution

\[ f(u_1, u_2) = \min\{u_1, u_2\} \]
Bargaining Solutions

• Nash Bargaining Solution
  • Example
  • Axiomatic justification
  • Bargaining justification

• Raiffa-Kalai-Smorodinsky Solution
  • Example
  • Axiomatic justification
  • Bargaining justification
Bargaining Solutions

• A **bargaining solution** is an equilibrium allocation in the sense that none of the parties wish to bargain further.

  • Because all parties are “satisfied” in some sense, the outcome may be viewed as “fair.”

  • Bargaining models have a **default** outcome, which is the result of a failure to reach agreement.

  • The default outcome can be seen as a **starting point**.
Bargaining Solutions

• Several proposals for the default outcome (starting point):

  • **Zero** for everyone. Useful when only the resources being allocated are relevant to fairness of allocation.

  • **Equal split.** Resources (not necessarily utilities) are divided equally. May be regarded as a “fair” **starting point**.

  • **Strongly pareto set.** Each party receives resources that can benefit no one else. Parties can always agree on this.
Nash Bargaining Solution

• The **Nash bargaining solution** maximizes the social welfare function

\[ f(u) = \prod_i (u_i - d_i) \]

where \( d \) is the default outcome.

• **Not** the same as **Nash equilibrium**.
• It maximizes the **product of the gains** achieved by the bargainers, relative to the fallback position.
• Assume feasible set is **convex**, so that Nash solution is unique (due to strict concavity of \( f \)).
Nash Bargaining Solution

Nash solution maximizes area of rectangle

Feasible set
Nash Bargaining Solution

Nash solution maximizes area of rectangle

Feasible set
Nash Bargaining Solution

Nash solution maximizes area of rectangle

Feasible set
Nash Bargaining Solution

• Major application to telecommunications.
  • Where it is known as proportional fairness
  • $u$ is proportionally fair if for all feasible allocations $u'$
    \[
    \sum_i \frac{u'_i - u_i}{u_i} \leq 0
    \]
  • Here, $u_i$ is the utility of the packet flow rate assigned user $i$.
  • Maximin criterion also used.
Nash Bargaining Solution

- The **optimization problem** has a concave objective function if we maximize $\log f(u)$.

$$\max \log \prod_i (u_i - d_i) = \sum_i \log (u_i - d_i)$$

$u \in S$

- Problem is relatively easy if feasible set $S$ is convex.
Nash Bargaining Solution
From Zero

\begin{align*}
\text{Graph showing Nash Bargaining Solution from zero.}
\end{align*}
Nash Bargaining Solution
From Equality

\[ (900, 750) \]
\[ (600, 600) \]
Nash Bargaining Solution

• **Strongly pareto set** gives Smith all 12 avocados.
  • Nothing for Jones.
  • Results in utility \((u_1, u_2) = (0,600)\)

Utility provided by one fruit of each kind

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</table>
Nash Bargaining Solution
From Strongly Pareto Set
**Axiomatic Justification**

- **Axiom 1.** Invariance under translation and rescaling.
  - If we map \( u_i \to a_i u_i + b_i \), \( d_i \to a_i d_i + b_i \),
    then bargaining solution \( u_i^* \to a_i u_i^* + b_i \).

This is **cardinal noncomparability**.
Axiomatic Justification

• **Axiom 1.** Invariance under translation and rescaling.
  - If we map \( u_i \rightarrow a_i u_i + b_i, \ d_i \rightarrow a_i d_i + b_i, \)
    then bargaining solution \( u_i^* \rightarrow a_i u_i^* + b_i. \)

• **Strong assumption** – failed, e.g., by utilitarian welfare function
Axiomatic Justification

• **Axiom 2.** Pareto optimality.
  • Bargaining solution is pareto optimal.

• **Axiom 3.** Symmetry.
  • If all $d_j$s are equal and feasible set is symmetric, then all $u_i^*$s are equal in bargaining solution.
Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow’s axiom.
  - If $u^*$ is a solution with respect to $d$
Axiomatic Justification

• **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow’s axiom.
  - If $u^*$ is a solution with respect to $d$, then it is a solution in a smaller feasible set that contains $u^*$ and $d$. 
Axiomatic Justification

**Axiom 4.** Independence of irrelevant alternatives.

- Not the same as Arrow’s axiom.
- If $u^*$ is a solution with respect to $d$, then it is a solution in a smaller feasible set that contains $u^*$ and $d$.
- This basically says that the solution behaves like an optimum.
Axiomatic Justification

**Theorem.** Exactly one solution satisfies Axioms 1-4, namely the Nash bargaining solution.

**Proof** (2 dimensions).

First show that the Nash solution satisfies the axioms.

**Axiom 1.** Invariance under transformation. If

\[ \prod_i (u_i^* - d_i) \geq \prod_i (u_i - d_i) \]

then

\[ \prod_i ((a_i u_i^* + b_i) - (a_i d_i + b_i)) \geq \prod_i ((a_i u_i + b_i) - (a_i d_i + b_i)) \]
Axiomatic Justification

**Axiom 2.** Pareto optimality. Clear because social welfare function is strictly monotone increasing.

**Axiom 3.** Symmetry. Obvious.

**Axiom 4.** Independence of irrelevant alternatives. Follows from the fact that $u^*$ is an optimum.

Now show that **only** the Nash solution satisfies the axioms.
Axiomatic Justification

Let $u^*$ be the Nash solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(u_1, u_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution $(1,1)$, by Axiom 1:
Axiomatic Justification

Let $u^*$ be the Nash solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(u_1,u_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution $(1,1)$, by Axiom 1:

By Axioms 2 & 3, $(1,1)$ is the only bargaining solution in the triangle:
Axiomatic Justification

Let $u^*$ be the Nash solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(u_1,u_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution $(1,1)$, by Axiom 1:

By Axioms 2 & 3, $(1,1)$ is the **only** bargaining solution in the triangle:

So by Axiom 4, $(1,1)$ is the only bargaining solution in blue set.
**Axiomatic Justification**

Let $u^*$ be the Nash solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(u_1, u_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution $(1,1)$, by Axiom 1:

So by Axiom 4, $(1,1)$ is the only bargaining solution in blue set.

By Axiom 1, $u^*$ is the only bargaining solution in the original problem.
Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** (invariance under transformation) is very strong.
  - Axiom 1 denies *interpersonal comparability*.
  - So how can it reflect moral concerns?
Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** (invariance under transformation) is very strong.
  - Axiom 1 denies *interpersonal comparability*.
  - So how can it reflect moral concerns?

- Most attention has been focused on **Axiom 4**
  (independence of irrelevant alternatives).
  - Will address this later.
Bargaining Justification

Players 1 and 2 make offers $s$, $t$. 
Bargaining Justification

Players 1 and 2 make offers $s$, $t$. Let $p = P$(player 2 will reject $s$), as estimated by player 1.
Bargaining Justification

Players 1 and 2 make offers $s$, $t$.

Let $p = P(\text{player 2 will reject } s)$, as estimated by player 1.

Then player 1 will stick with $s$, rather than make a counteroffer, if

$$(1 - p)s + pd \geq t$$
Bargaining Justification

Players 1 and 2 make offers \( s, t \).
Let \( p = P(\text{player 2 will reject } s) \), as estimated by player 1.
Then player 1 will stick with \( s \), rather than make a counteroffer, if

\[
(1 - p)s_1 + pd_1 \geq t_1
\]

So player 1 will stick with \( s \) if

\[
p \leq \frac{s_1 - t_1}{s_1 - d_1} = r_1
\]
Bargaining Justification

It is rational for player 1 to make a counteroffer $s'$, rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$

So player 1 will stick with $s$ if

$$p \leq \frac{s_1 - t_1}{s_1 - d_1} = r_1$$
Bargaining Justification

It is rational for player 1 to make a counteroffer $s'$, rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$

It is rational for player 2 to make the next counteroffer if

$$r'_1 = \frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2$$
Bargaining Justification

It is rational for player 1 to make a counteroffer $s'$, rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$

It is rational for player 2 to make the next counteroffer if

$$r'_1 = \frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2$$

But

$$\frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2}$$
Bargaining Justification

It is rational for player 1 to make a counteroffer $s'$, rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$

It is rational for player 2 to make the next counteroffer if

$$r'_1 = \frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2$$

But

$$\frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} \iff \frac{t_1 - d_1}{s_1 - d_1} \geq \frac{s_2 - d_2}{t_2 - d_2}$$
Bargaining Justification

So we have \( (s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2) \)

It is rational for player 2 to make the next counteroffer if

\[ r_1' = \frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2}{t_2 - d_2} = r_2' \]

But

\[ \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} \]

\[ \iff \frac{t_1 - d_1}{s_1 - d_1} \geq \frac{s_2 - d_2}{t_2 - d_2} \]
Bargaining Justification

So we have \((s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\)

It is rational for player 2 to make the next counteroffer if

\[
r_1' = \frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2} = r_2'
\]

Similarly \(\frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2}\)
Bargaining Justification

So we have \[(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\]

It is rational for player 2 to make the next counteroffer if
\[r'_1 = \frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2\]

Similarly
\[\frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2}\]

\[\frac{t_1 - d_1}{s'_1 - d_1} \leq \frac{s'_2 - d_2}{t_2 - d_2}\]
Bargaining Justification

So we have \((s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\)
and we have \((t_1 - d_1)(t_2 - d_2) \leq (s'_1 - d_1)(s'_2 - d_2)\)

Similarly \[\frac{s' - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2}\]
\[\frac{t_1 - d_1}{s'_1 - d_1} \leq \frac{s'_2 - d_2}{t_2 - d_2}\]
Bargaining Justification

So we have \[(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\]
and we have \[(t_1 - d_1)(t_2 - d_2) \leq (s'_1 - d_1)(s'_2 - d_2)\]

This implies an improvement in the Nash social welfare function
Bargaining Justification

So we have \((s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\)
and we have \((t_1 - d_1)(t_2 - d_2) \leq (s_1' - d_1)(s_2' - d_2)\)

This implies an improvement in the Nash social welfare function.

Given a minimum distance between offers, continued bargaining converges to Nash solution.
Raiffa-Kalai-Smorodinsky Bargaining Solution

- This approach begins with a critique of the Nash bargaining solution.
Raiffa-Kalai-Smorodinsky Bargaining Solution

• This approach begins with a critique of the Nash bargaining solution.
  • The new Nash solution is worse for player 2 even though the feasible set is larger.
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from $d$ to ideal solution.
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from $d$ to ideal solution.
  - The players receive an equal fraction of their possible utility gains.

\[
\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}
\]
Raiffa-Kalai-Smorodinsky Bargaining Solution

**Proposal**: Bargaining solution is pareto optimal point on line from $d$ to ideal solution.

- Replace Axiom 4 with **Axiom 4’ (Monotonicity)**: A larger feasible set with same ideal solution results in a bargaining solution that is better (or no worse) for all players.

\[ \frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2} \]
Raiffa-Kalai-Smorodinsky Bargaining Solution

• **Applications**
  - Allocation of wireless capacity.
  - Allocation of cloud computing resources.
  - Datacenter resource scheduling
    (also dominant resource fairness)
  - Resource allocation in visual sensor networks
  - Labor-market negotiations
Raiffa-Kalai-Smorodinsky Bargaining Solution

• **Optimization model.**
  • Not an optimization problem over original feasible set (we gave up Axiom 4).
  • But it is an optimization problem (pareto optimality) over the line segment from \( d \) to ideal solution.

\[
\max \sum_i u_i \\
(g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_i - d_1), \quad \text{all } i \]
\[
u \in S
\]

\[
\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}
\]
Raiffa-Kalai-Smorodinsky Bargaining Solution

• **Optimization model.**
  • Not an optimization problem over original feasible set (we gave up Axiom 4).
  • But it is an optimization problem (pareto optimality) over the line segment from \( d \) to ideal solution.

\[
\max \sum_i u_i
\]
\[
(g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_1 - d_1), \text{ all } i
\]
\[
u \in S
\]
Raiffa-Kalai-Smorodinskyky Bargaining Solution

- **Optimization model.**
  - Not an optimization problem over original feasible set (we gave up Axiom 4).
  - But it is an optimization problem (pareto optimality) over the line segment from $d$ to ideal solution.

\[
\max \sum_{i} u_i \quad \text{subject to} \quad (g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_1 - d_1), \quad \text{all } i
\]

\[u \in S\]

**Linear constraint**
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Zero

\[ (0,0) \rightarrow (800,800) \]
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Equality

\[(900,750), (600,600)\]
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Strong Pareto Set

\[ (600, 900) \]

\[ (0, 600) \]
Axiomatic Justification

- **Axiom 1.** Invariance under transformation.
- **Axiom 2.** Pareto optimality.
- **Axiom 3.** Symmetry.
- **Axiom 4′.** Monotonicity.
Axiomatic Justification

**Theorem.** Exactly one solution satisfies Axioms 1-4', namely the RKS bargaining solution.

**Proof** (2 dimensions).

Easy to show that RKS solution satisfies the axioms.

Now show that **only** the RKS solution satisfies the axioms.
Axiomatic Justification

Let $u^*$ be the RKS solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(g_1,g_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has RKS solution $u'$, by Axiom 1:
Axiomatic Justification

Let $u^*$ be the RKS solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(g_1,g_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

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By Axioms 2 & 3, $u'$ is the **only** bargaining solution in the red polygon:
Axiomatic Justification

Let $u^*$ be the RKS solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends
\[(g_1, g_2) \rightarrow (1, 1), \quad (d_1, d_2) \rightarrow (0, 0)\]
The transformed problem has RKS solution $u'$, by Axiom 1:

By Axioms 2 & 3, $u'$ is the only bargaining solution in the red polygon:

The red polygon lies inside blue set. So by Axiom 4′, its bargaining solution is no better than bargaining solution on blue set. So $u'$ is the only bargaining solution on blue set.
Axiomatic Justification

Let $u^*$ be the RKS solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(g_1, g_2) \rightarrow (1, 1), \quad (d_1, d_2) \rightarrow (0, 0)$$

The transformed problem has RKS solution $u'$, by Axiom 1:

By Axiom 1, $u^*$ is the only bargaining solution in the original problem.
Axiomatic Justification

• **Problems** with axiomatic justification.
  • **Axiom 1** is still in effect.
  • It denies **interpersonal comparability**.
  • Dropping Axiom 4 sacrifices optimization of a social welfare function.
  • This may not be necessary if Axiom 1 is rejected.
  • Needs modification for > 2 players (more on this shortly).
Bargaining Justification

Resistance to an agreement $s$ depends on sacrifice relative to sacrifice under no agreement. Here, player 2 is making a larger relative sacrifice:

$$\frac{g_1 - s_1}{g_1 - d_1} \leq \frac{g_2 - s_2}{g_2 - d_2}$$

Minimizing resistance to agreement requires minimizing

$$\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}$$
Bargaining Justification

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or equivalently, maximizing

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Bargaining Justification

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\[
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\]

Minimizing resistance to agreement requires minimizing

\[
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\]

or equivalently, maximizing

\[
\min_i \left\{ \frac{s_i - d_i}{g_i - d_i} \right\}
\]

which is achieved by RKS point.
Bargaining Justification

This is the **Rawlsian social contract** argument applied to **gains** relative to the ideal.

Minimizing resistance to agreement requires minimizing

\[
\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}
\]

or equivalently, maximizing

\[
\min_i \left\{ \frac{s_i - d_i}{g_i - d_i} \right\}
\]

which is achieved by RKS point.
Problem with RKS Solution

- However, the RKS solution is Rawlsian only for 2 players.
  - In fact, RKS leads to counterintuitive results for 3 players.

Red triangle is feasible set.
RKS point is $d$!
Problem with RKS Solution

• However, the RKS solution is Rawlsian only for 2 players.
  • In fact, RKS leads to counterintuitive results for 3 players.

Red triangle is feasible set.
RKS point is \( d \)!
Rawlsian point is \( u \).
Summary
Summary

![Graph showing Nash bargaining and Utilitarian approaches with points (0,600), (600,600), and (0,0)]
Summary

Raiffa-Kalai-Smorodinsky bargaining
Nash bargaining
Utilitarian

u_2

(0,600) (600,600) (0,0)

u_1

1200

Na
Combining Equity and Efficiency

- A proposed model
- Health care application
Combining Equity and Efficiency

• Utilitarian and Rawlsian distributions seem too extreme in practice.
  – How to combine them?

• One proposal:
  – Maximize welfare of worst off (Rawlsian)...
  – until this requires undue sacrifice from others
  – Seems appropriate in health care allocation.
Combining Equity and Efficiency

• In particular:
  
  – Switch from Rawlsian to utilitarian when inequality exceeds $\Delta$. 
Combining Equity and Efficiency

• In particular:
  – Switch from Rawlsian to utilitarian when inequality exceeds $\Delta$.
  – Build mixed integer programming model.
  – Let $u_i$ = utility allocated to person $i$

• For 2 persons:
  – Maximize $\min_i \{u_1, u_2\}$ (Rawlsian) when $|u_1 - u_2| \leq \Delta$
  – Maximize $u_1 + u_2$ (utilitarian) when $|u_1 - u_2| > \Delta$
Two-person Model

Contours of **social welfare function** for 2 persons.
Two-person Model

Contours of social welfare function for 2 persons.

Rawlsian region \( \min\{u_1, u_2\} \)
Two-person Model

Contours of **social welfare function** for 2 persons.

Utilitarian region

\[ u_1 + u_2 \]

Rawlsian region

\[ \min\{u_1, u_2\} \]
Person 1 is harder to treat.

But maximizing person 1’s health requires too much sacrifice from person 2.
Advantages

• Only one parameter $\Delta$
  – Focus for debate.
  – $\Delta$ has intuitive meaning (unlike weights)
  – Examine consequences of different settings for $\Delta$
  – Find least objectionable setting
  – Results in a consistent policy
We want continuous contours
Social Welfare Function

We want continuous contours

\[ u_1 + u_2 \]

\[ 2\min\{u_1, u_2\} + \Delta \]

So we use affine transform of Rawlsian criterion
Social Welfare Function

The social welfare problem becomes

\[
\begin{align*}
\max & \quad z \\
\text{subject to} & \quad z \leq \begin{cases} 
2 \min \{u_1, u_2\} + \Delta, & \text{if } |u_1 - u_2| \leq \Delta \\
u_1 + u_2, & \text{otherwise}
\end{cases}
\end{align*}
\]

constraints on feasible set
MILP Model

Epigraph is union of 2 polyhedra.
MILP Model

Epigraph is union of 2 polyhedra. Because they have **different recession cones**, there is no MILP model.
MILP Model

Impose constraints $|u_1 - u_2| \leq M$
MILP Model

This equalizes recession cones.

Recession directions \((u_1, u_2, z)\)
MILP Model

We have the model

\[
\begin{align*}
\text{max} & \quad z \\
z & \leq 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1, 2 \\
z & \leq u_1 + u_2 + \Delta(1 - \delta) \\
u_1 - u_2 & \leq M, \quad u_2 - u_1 \leq M \\
u_1, u_2 & \geq 0 \\
\delta & \in \{0, 1\} \\
\text{constraints on feasible set} & \quad u_1
\end{align*}
\]
MILP Model

We have the model

\[ \begin{align*}
\max \ z \\
z &\leq 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1,2 \\
z &\leq u_1 + u_2 + \Delta(1 - \delta) \\
u_1 - u_2 &\leq M, \quad u_2 - u_1 \leq M \\
u_1, u_2 &\geq 0 \\
\delta &\in \{0,1\}
\end{align*} \]

This is a convex hull formulation.
$n$-person Model

Rewrite the 2-person social welfare function as

$$\Delta + 2u_{\min} + (u_1 - u_{\min} - \Delta)^+ + (u_2 - u_{\min} - \Delta)^+$$

$$\min\{u_1, u_2\}$$

$$\alpha^+ = \max\{0, \alpha\}$$
\textit{n-person Model}

Rewrite the 2-person social welfare function as

\[
\Delta + 2u_{\min} + (u_1 - u_{\min} - \Delta)^+ + (u_2 - u_{\min} - \Delta)^+ \\
\min\{u_1, u_2\} \quad \alpha^+ = \max\{0, \alpha\}
\]

This can be generalized to \(n\) persons:

\[
(n - 1)\Delta + nu_{\min} + \sum_{j=1}^{n} (u_j - u_{\min} - \Delta)^+ 
\]
**n-person Model**

Rewrite the 2-person social welfare function as

\[
\Delta + 2u_{\text{min}} + (u_1 - u_{\text{min}} - \Delta)^+ + (u_2 - u_{\text{min}} - \Delta)^+ \\
\min\{u_1,u_2\} \quad \alpha^+ = \max\{0,\alpha\}
\]

This can be generalized to \( n \) persons:

\[
(n-1)\Delta + nu_{\text{min}} + \sum_{j=1}^{n} (u_j - u_{\text{min}} - \Delta)^+ 
\]

Epigraph is a union of \( n! \) polyhedra with same recession direction \((u,z) = (1, ,1,n)\) if we require \(|u_i - u_j| \leq M\)

So there is an MILP model
\textbf{\textit{n}-person MILP Model}

To avoid \(n!\) 0-1 variables, add auxiliary variables \(w_{ij}\)

\[
\begin{align*}
\text{max} & \quad z \\
z & \leq u_i + \sum_{j \neq i} w_{ij}, \text{ all } i \\
w_{ij} & \leq \Delta + u_i + \delta_{ij}(M - \Delta), \text{ all } i, j \text{ with } i \neq j \\
w_{ij} & \leq u_j + (1 - \delta_{ij})\Delta, \text{ all } i, j \text{ with } i \neq j \\
u_i - u_j & \leq M, \text{ all } i, j \\
u_i & \geq 0, \text{ all } i \\
\delta_{ij} & \in \{0,1\}, \text{ all } i, j \text{ with } i \neq j
\end{align*}
\]
**n-person MILP Model**

To avoid $n!$ 0-1 variables, add auxiliary variables $w_{ij}$

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\text{max} & \quad z \\
z & \leq u_i + \sum_{j \neq i} w_{ij}, \text{ all } i \\
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w_{ij} & \leq u_j + (1 - \delta_{ij}) \Delta, \text{ all } i, j \text{ with } i \neq j \\
u_i - u_j & \leq M, \text{ all } i, j \\
u_i & \geq 0, \text{ all } i \\
\delta_{ij} & \in \{0, 1\}, \text{ all } i, j \text{ with } i \neq j
\end{align*}$$

**Theorem.** The model is correct (not easy to prove).
 \textbf{n-person MILP Model}

To avoid \(n!\) 0-1 variables, add auxiliary variables \(w_{ij}\)

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\begin{align*}
\text{max} & \quad z \\
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   w_{ij} & \leq \Delta + u_i + \delta_{ij}(\mathcal{M} - \Delta), \text{ all } i, j \text{ with } i \neq j \\
   w_{ij} & \leq u_j + (1 - \delta_{ij})\Delta, \text{ all } i, j \text{ with } i \neq j \\
   u_i - u_j & \leq \mathcal{M}, \text{ all } i, j \\
   u_i & \geq 0, \text{ all } i \\
   \delta_{ij} & \in \{0,1\}, \text{ all } i, j \text{ with } i \neq j
\end{align*}
\]

\textbf{Theorem.} The model is correct (not easy to prove).

\textbf{Theorem.} This is a convex hull formulation (not easy to prove).
$n$-group Model

In practice, funds may be allocated to groups of different sizes.

For example, disease/treatment categories.

Let $\bar{u}_i = $ average utility gained by a person in group $i$

$n_i = $ size of group $i$
$n$-group Model

2-person case with $n_1 < n_2$. Contours have slope $-n_1/n_2$.
\textbf{n-group MILP Model}

Again add auxiliary variables $w_{ij}$

\[
\begin{align*}
\text{max} \; z \\
z & \leq (n_i - 1)\Delta + n_i\bar{u}_i + \sum_{j \neq i} w_{ij}, \; \text{all } i \\
w_{ij} & \leq n_j(\bar{u}_i + \Delta) + \delta_{ij} n_j(M - \Delta), \; \text{all } i, j \text{ with } i \neq j \\
w_{ij} & \leq \bar{u}_j + (1 - \delta_{ij})n_j\Delta, \; \text{all } i, j \text{ with } i \neq j \\
\bar{u}_i - \bar{u}_j & \leq M, \; \text{all } i, j \\
\bar{u}_i & \geq 0, \; \text{all } i \\
\delta_{ij} & \in \{0, 1\}, \; \text{all } i, j \text{ with } i \neq j
\end{align*}
\]

\textbf{Theorem.} The model is correct.

\textbf{Theorem.} This is a convex hull formulation.
Health Example

Measure utility in QALYs (quality-adjusted life years).

QALY and cost data based on Briggs & Gray, (2000) etc.

Each group is a disease/treatment pair.

Treatments are discrete, so group funding is all-or-nothing.

Divide groups into relatively homogeneous subgroups.
Health Example

Add constraints to define feasible set

\[
\begin{align*}
\text{max } & \quad z \\
& \quad z \leq (n_i - 1)\Delta + n_i\bar{u}_i + \sum_{j \neq i} w_{ij}, \text{ all } i \\
& \quad w_{ij} \leq n_j(\bar{u}_i + \Delta) + \delta_{ij}n_j(M - \Delta), \text{ all } i, j \text{ with } i \neq j \\
& \quad w_{ij} \leq \bar{u}_j + (1 - \delta_{ij})n_j\Delta, \text{ all } i, j \text{ with } i \neq j \\
& \quad \bar{u}_i - \bar{u}_j \leq M, \text{ all } i, j \\
& \quad \bar{u}_i \geq 0, \text{ all } i \\
& \quad \delta_{ij} \in \{0, 1\}, \text{ all } i, j \text{ with } i \neq j \\
\end{align*}
\]

\[\bar{u}_i = a_i y_i + \alpha_i\]

\[\sum_i n_i c_i y_i \leq \text{ budget}\]

\[y_i \in \{0, 1\}, \text{ all } i\]
<table>
<thead>
<tr>
<th>Intervention</th>
<th>Cost per person $c_i$ (£)</th>
<th>QALYs gained $q_i$</th>
<th>Cost per QALY $\alpha_i$ (£)</th>
<th>QALYs without intervention $\alpha_i$</th>
<th>Subgroup size $n_i$</th>
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## Results

**Utilitarian solution**

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More dialysis with larger Δ, beginning with longer life span
# Results

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Results

Most rapid change. Possible range for politically acceptable compromise

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# Results

32 groups, 1089 integer variables
Solution time (CPLEX 12.2) is negligible

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### Results

Table 3: Solution times in seconds for $m$ groups and different values of $\Delta$. Instances with more than a few hundred groups seem very unlikely to occur in practice.

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