Modeling Distributive Justice

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Just Distribution

- **The problem:** How to distribute resources…

Salaries

Tax breaks

Medical care

Education

Government benefits

Justice and Optimization

- The problem is not to satisfy preferences, but to **achieve justice.**

Justice and Optimization

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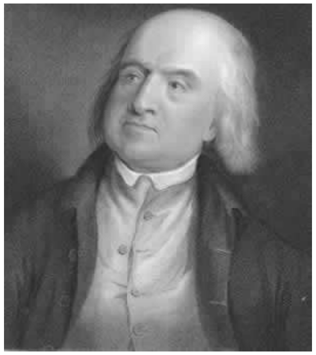
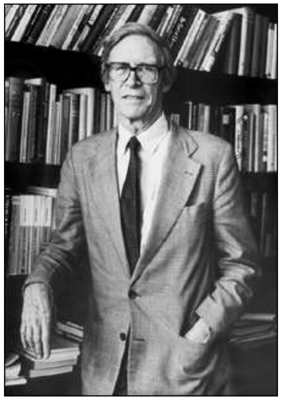
- Two classical criteria for distributive justice:
  - **Utilitarianism**
  - **Difference principle** of **John Rawls**





Justice and Optimization

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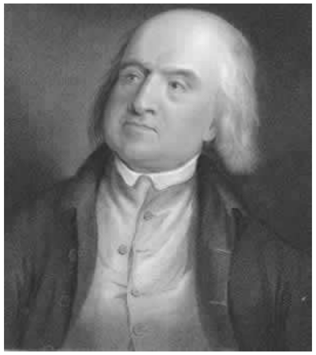

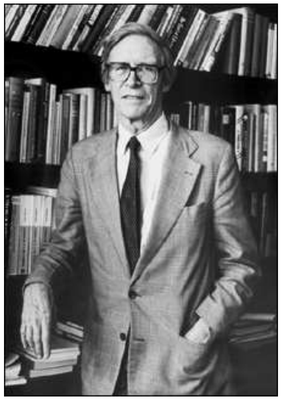

- Both can be viewed as **mathematical optimization problems.**

Justice and Optimization

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- **Utilitarianism** seeks allocation of wealth to individuals that maximizes total utility.
- The **Rawlsian difference principle** calls for a lexicographic maximum of utilities allotted to individuals**.**
- The two principles can also be **combined.**

Justice and Optimization

- We analyze distributions over **nonidentical individuals.**
  - Unlike most mathematical/axiomatic treatments of social welfare.

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  - i.e., to individuals who are **more talented** or **work harder**.

Justice and Optimization

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  - Unlike most mathematical/axiomatic treatments of social welfare.
- Distribution of greater resources to **more productive individuals** may increase overall utility.
  - i.e., to individuals who are **more talented** or **work harder**.
- To what extent does **efficiency require inequality** in the utilitarian and Rawlsian models?

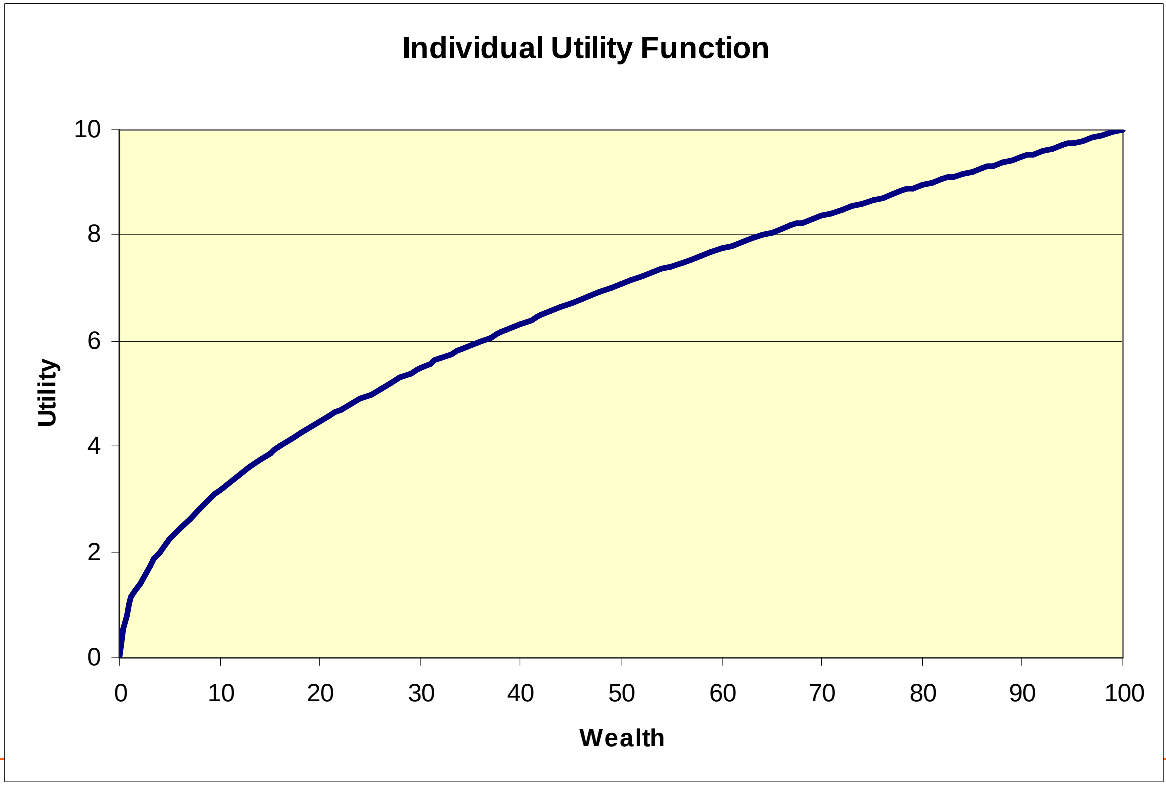
Outline

- **Utilitarian** principle
  - Optimality analysis
- **Difference** principle
  - Analysis of **lexmax model**
- A **combined** principle
  - Key application: **Health care**
  - **Mixed integer** model & example

Utilitarian Principle

Utilitarian Principle

- We assume that every individual has a utility function *v(x)*, where *x* is the wealth allocation to the individual.



Utilitarian Principle

- A "just" distribution of wealth is one that maximizes total expected utility.

- Let x_i = wealth initially allocated to person *i*
  $u_i(x_i)$ = utility eventually produced by person *I*

Utilitarian Model

- The utility maximization problem:

$$\max \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1$$

Total budget

$$x_i \geq 0, \quad \text{all } i$$

Utilitarian Model

- Elementary KKT analysis yields the optimal solution:

$$u_1'(x_1) = \cdots = u_n'(x_n)$$

Marginal productivity

Distribute wealth so as to equalize marginal productivity.

Utilitarian Model

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Marginal productivity

Distribute wealth so as to equalize marginal productivity.

- If we index persons in order of marginal productivity, i.e.,

$$u_i'(\cdot) \leq u_{i+1}'(\cdot), \quad \text{all } i$$

Then less productive individuals receive less wealth.

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Marginal productivity

Distribute wealth so as to equalize marginal productivity.

- If we index persons in order of marginal productivity, i.e.,

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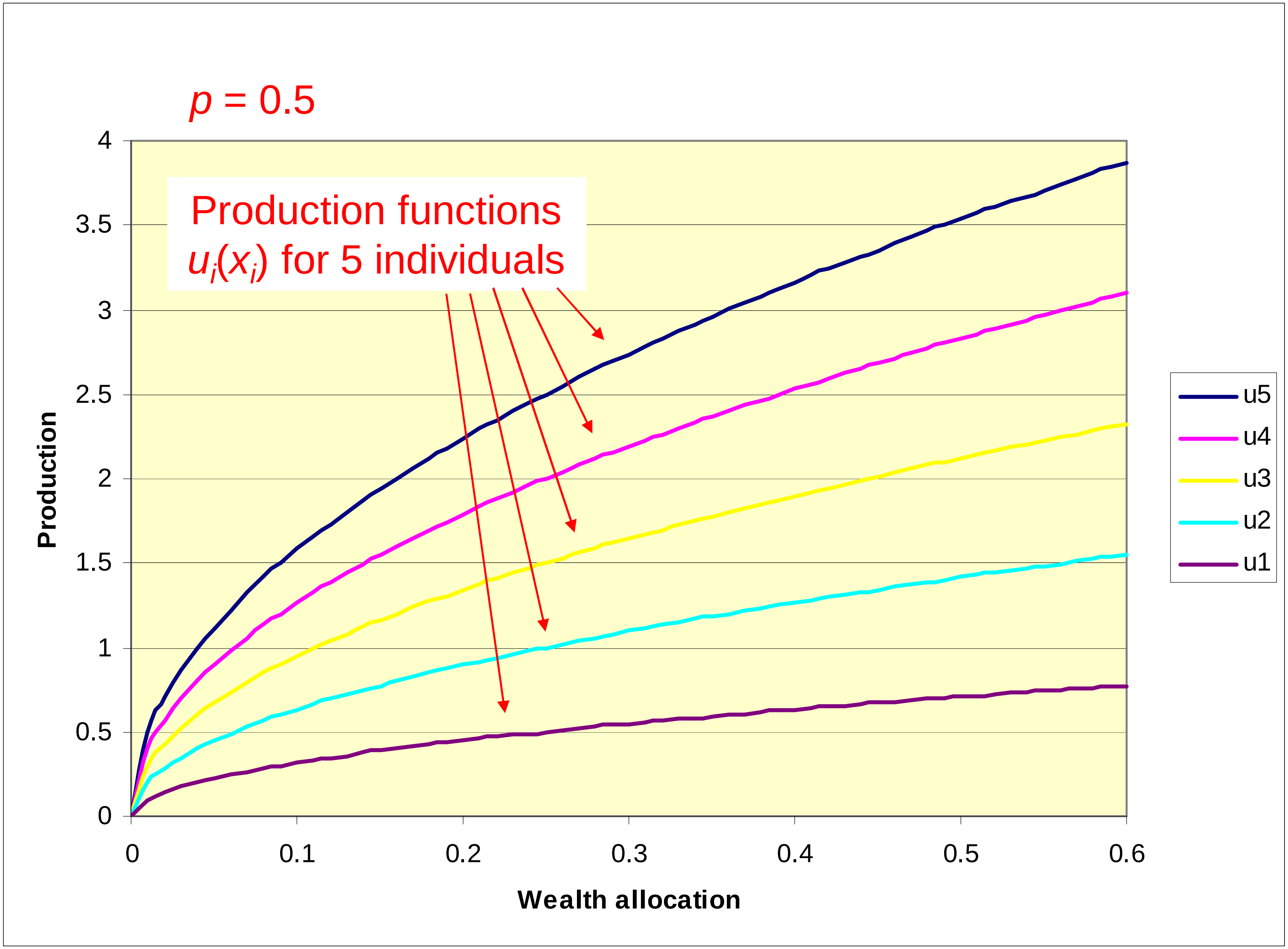
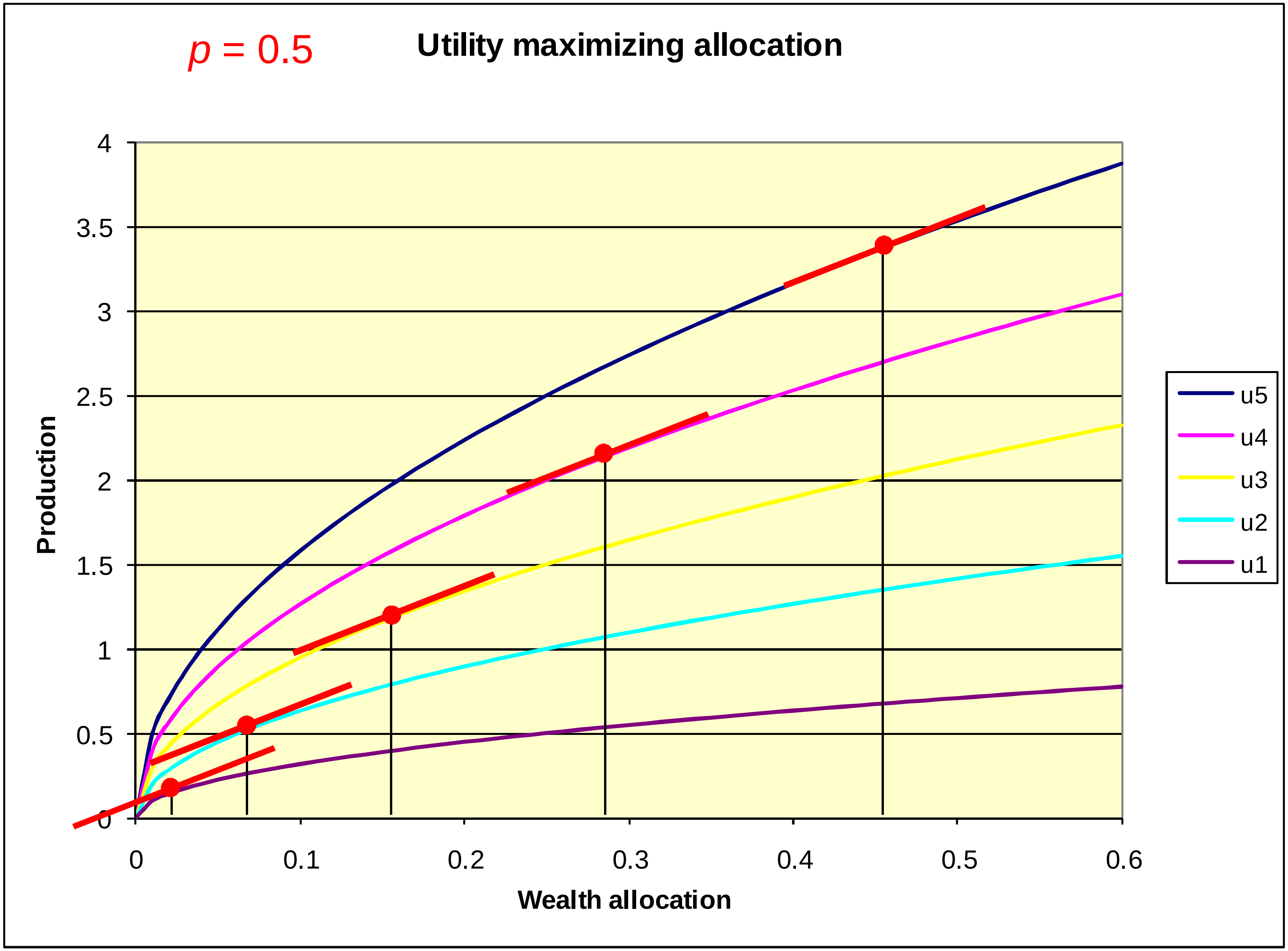
Then less productive individuals receive less wealth.

- For convenience assume $u_i(x_i) = c_i x_i^p$



p = 0.5
Production functions
$u_i(x_i)$ for 5 individuals
Production
Wealth allocation
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u5
u4
u3
u2
u1



p = 0.5
Utility maximizing allocation
Production
Wealth allocation
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
u5
u4
u3
u2
u1

Utilitarian Model

- Classical utilitarian argument: concave utility functions tend to make the utilitarian solution more **egalitarian**.

Utilitarian Model

- Classical utilitarian argument: concave utility functions tend to make the utilitarian solution more **egalitarian**.
- A **completely** egalitarian allocation $x_1 = \cdots = x_n$ is optimal only when

$$u_1'(1/n) = \cdots = u_n'(1/n)$$

- So, equality is optimal only when everyone has the same marginal productivity in an egalitarian allocation.

Utilitarian Model

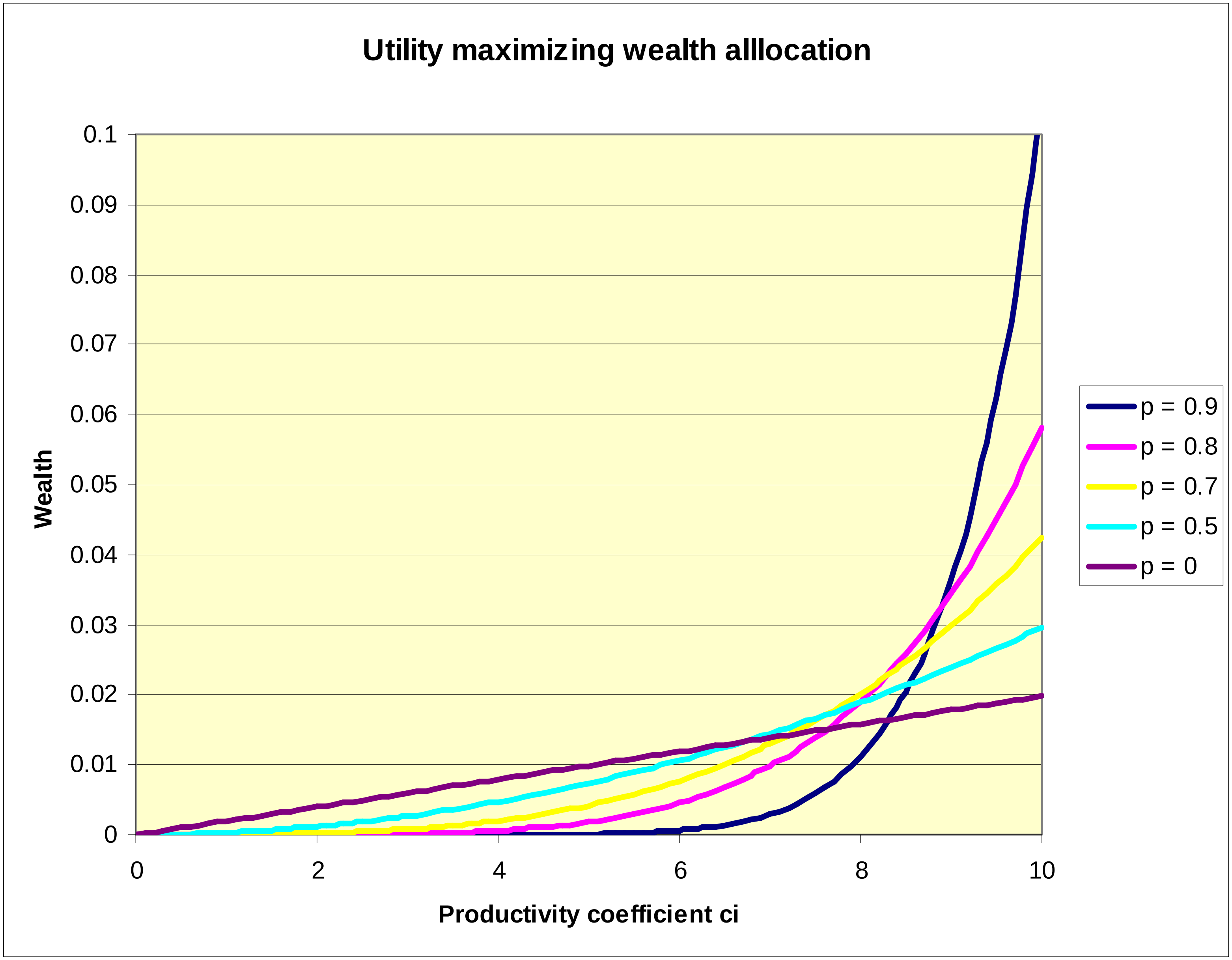
- Recall that $u_i(x_i) = c_i x_i^p$ where $p \geq 0$
- The optimal wealth allocation is

$$x_i = c_i^{\frac{1}{1-p}} \left(\sum_{j=1}^{n} c_j^{\frac{1}{1-p}} \right)^{-1}$$

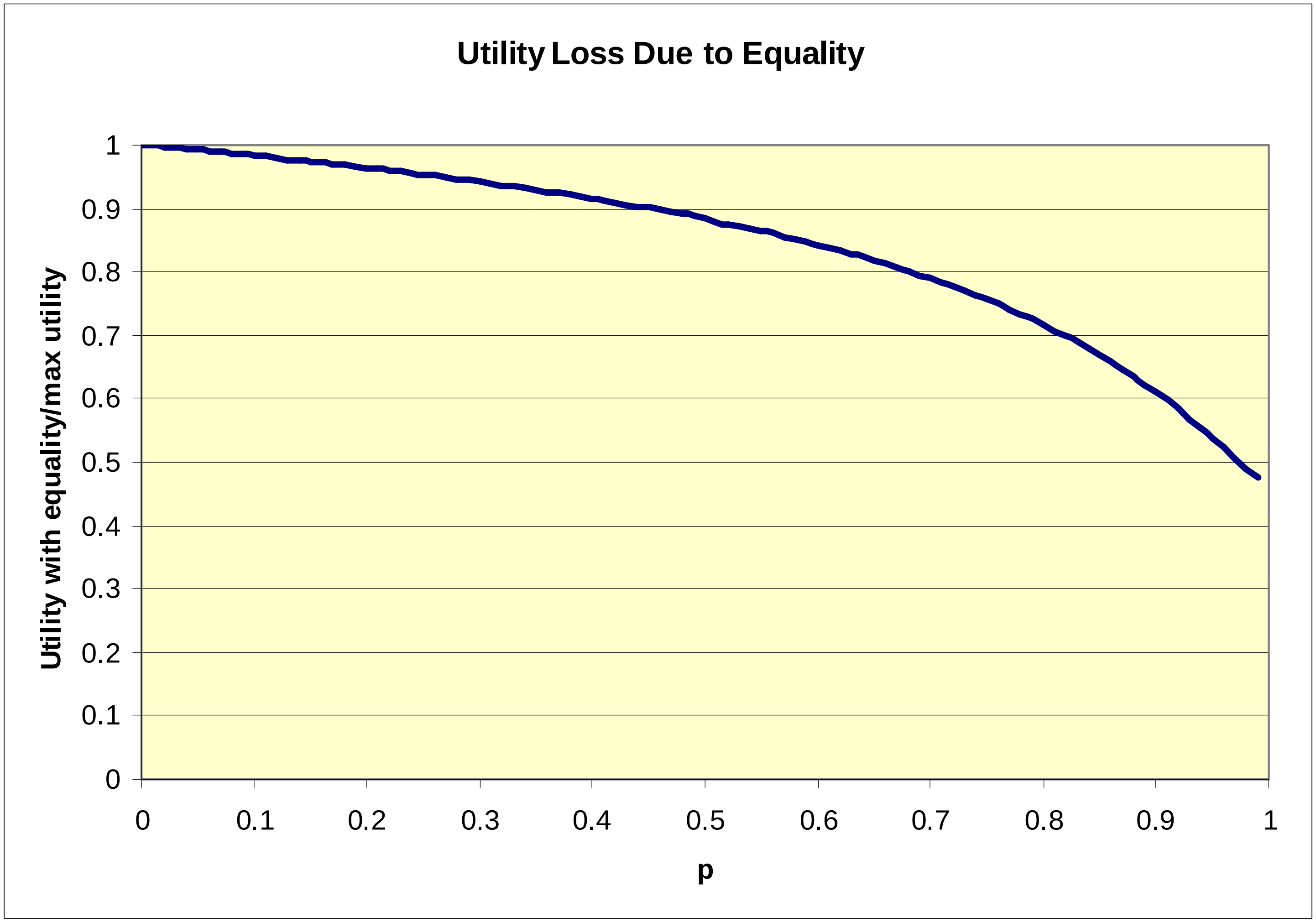
- When $p < 1$:
  - Allocation is **completely egalitarian** only if $c_1 = \cdots = c_n$
  - Otherwise the **most egalitarian** allocation occurs when $p \to 0$: $x_i = \dfrac{c_i}{\sum_j c_j}$

Utilitarian Model

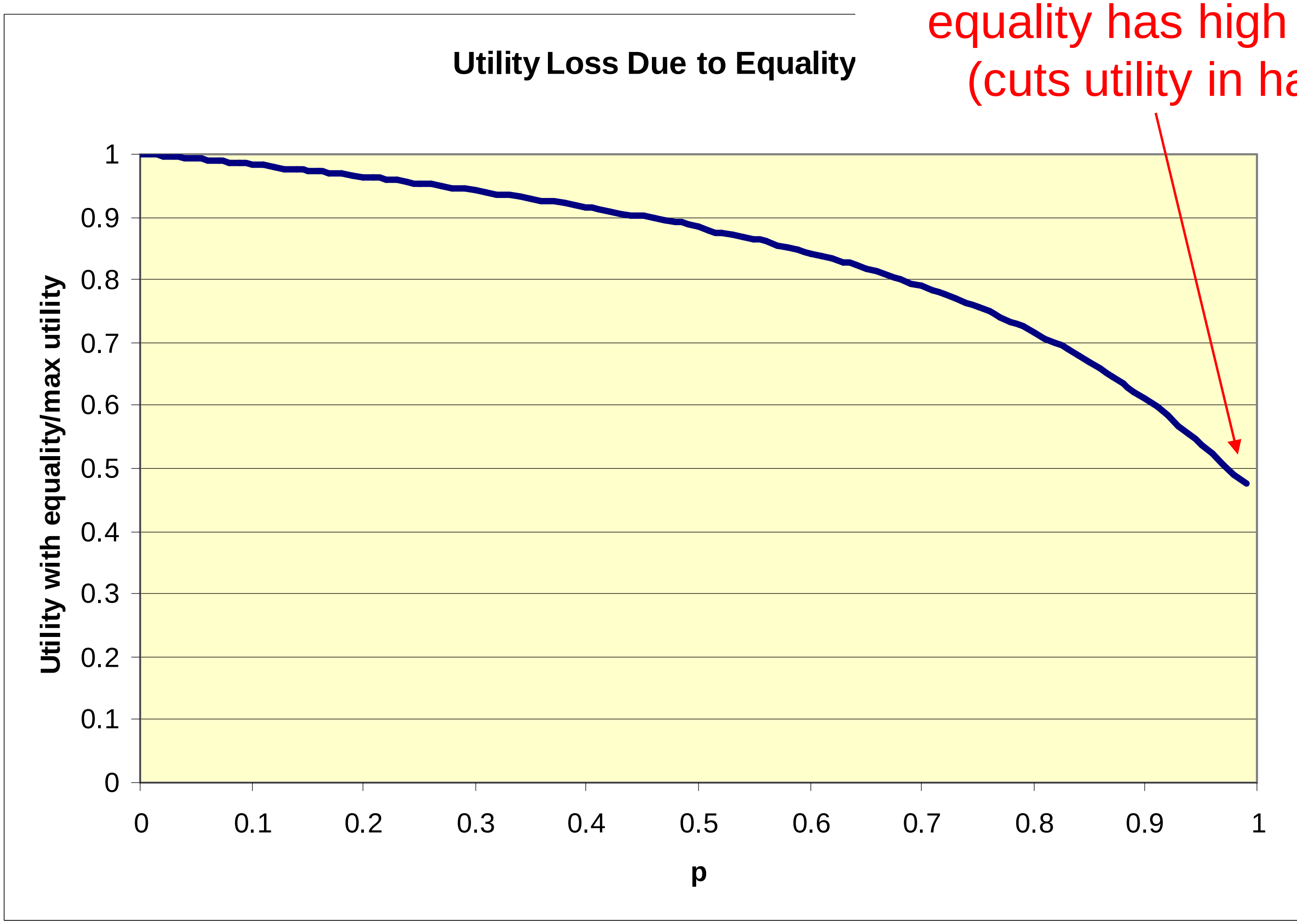
- The **most egalitarian** optimal allocation: people receive wealth in proportion to productivity c_i.
  - And this occurs only when productivity very insensitive to investment ($p \to 0$).



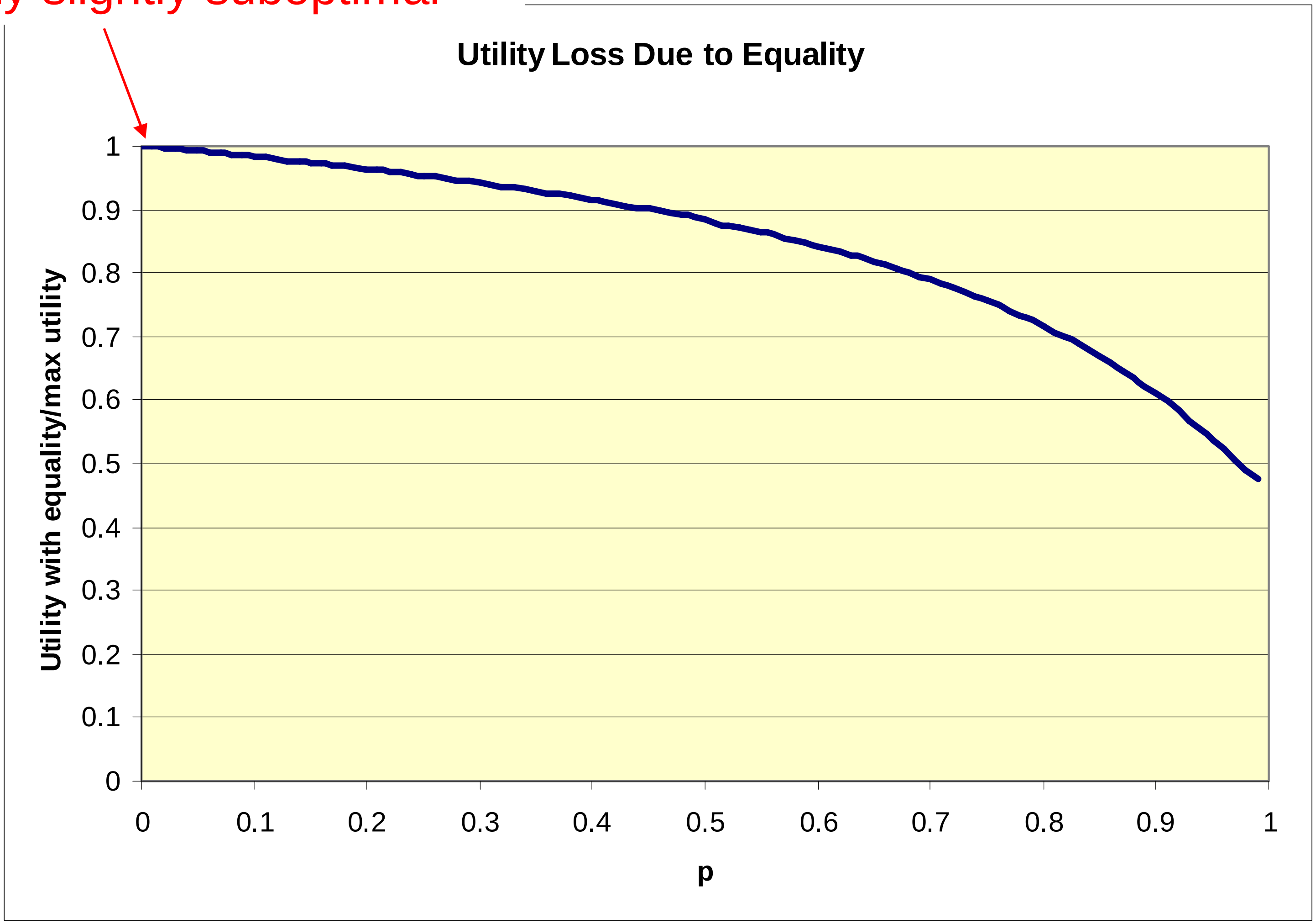
Utility maximizing wealth alllocation
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Wealth
0
2
4
6
8
10
Productivity coefficient ci
p = 0.9
p = 0.8
p = 0.7
p = 0.5
p = 0


Utility Loss Due to Equality
Utility with equality/max utility
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
p

When output is proportional to investment, equality has high cost (cuts utility in half)



As $p \to 0$, optimal utility requires highly unequal allocation, but equal allocation is only slightly suboptimal



Difference Principle

Problems with Utilitarianism

- A utility maximizing allocation may be **unjust**.
  - Disabled or disadvantaged people may be neglected.
  - Less talented people who work hard may receive meager wage.

Rawlsian Difference Principle

- **Difference principle:** A just distribution of wealth creates only as much inequality as is necessary to maximize the welfare of the worst off.
  - This refers to inequality of **opportunity**, not outcome.
  - Inequality may be necessary to incentivize everyone to **work harder** and therefore **raise the bottom**.

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- **Extension: lexmax (lexicographic maximum) principle**:
  - Maximize welfare of least advantaged class…
  - then next-to-least advantaged class…
  - and so forth.

Lexmax Model

- Assume each person's share of total utility is **proportional** to the utility of his/her initial wealth allocation.
  - Thus individuals with more education, salary have greater access to social utility.

- Assume productivity functions $u_i(x_i) = c_i x_i^p$
  - Larger p means productivity more sensitive to investment.

- Assume personal utility function $v(x_i) = x_i^q$
  - Larger q means people care more about getting rich.

Lexmax Model

- The utility maximization problem:

$$\text{lexmax } (y_1, \ldots, y_n)$$

$$\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1$$

Wealth allocation to person *i*

$$\boxed{x_i} \geq 0, \text{ all } i$$

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Budget

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yi's sum to total utility produced

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- The utility maximization problem:

Proportional allocation of total utility

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Utility allocation to person *i*

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\end{aligned}}$$

Theorem. If $u_i'(\cdot) \leq u_{i+1}'(\cdot)$ and $v(\cdot)$ is nondecreasing, this has an optimal solution in which $y_1 \leq \cdots \leq y_n$

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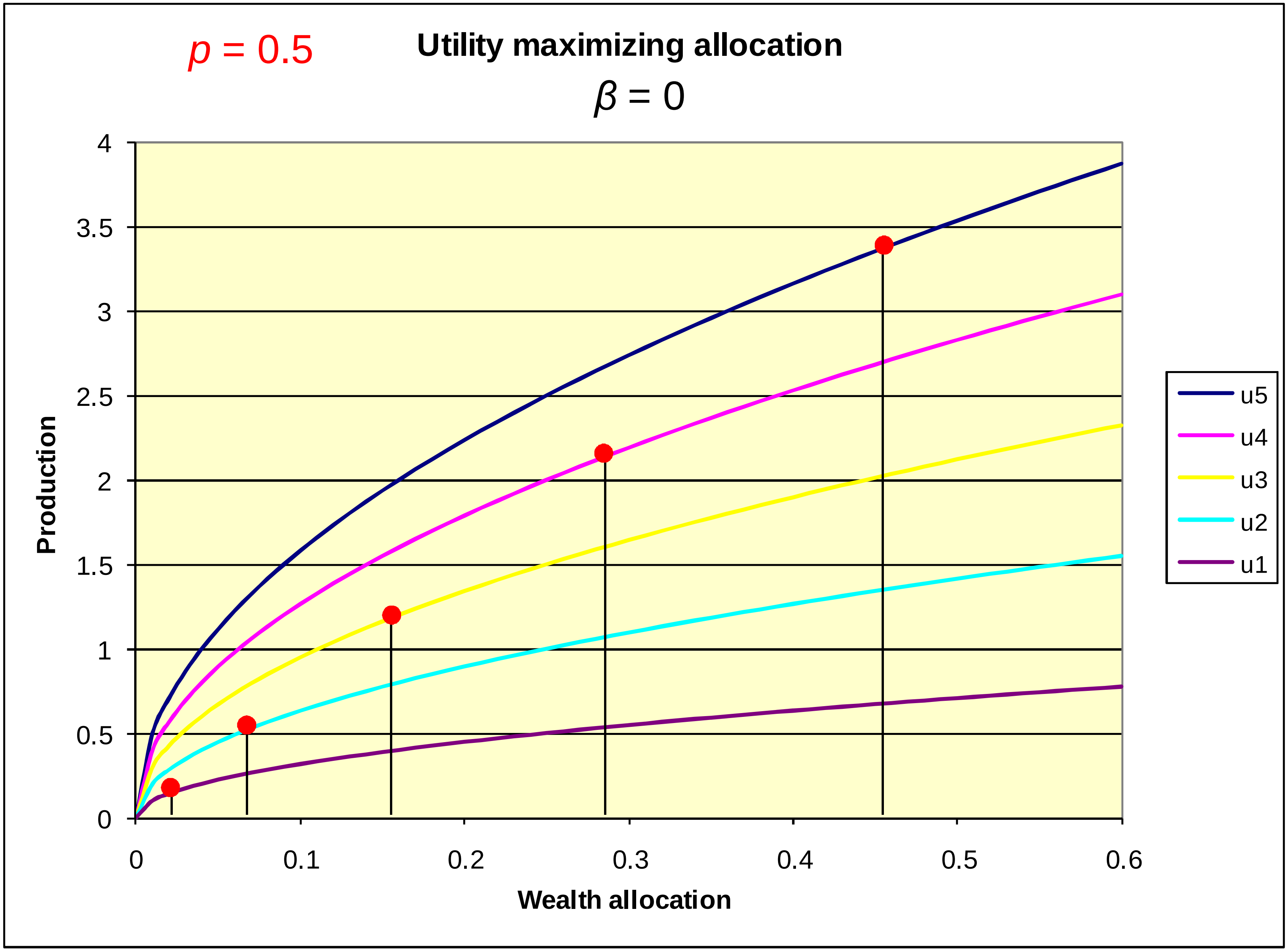
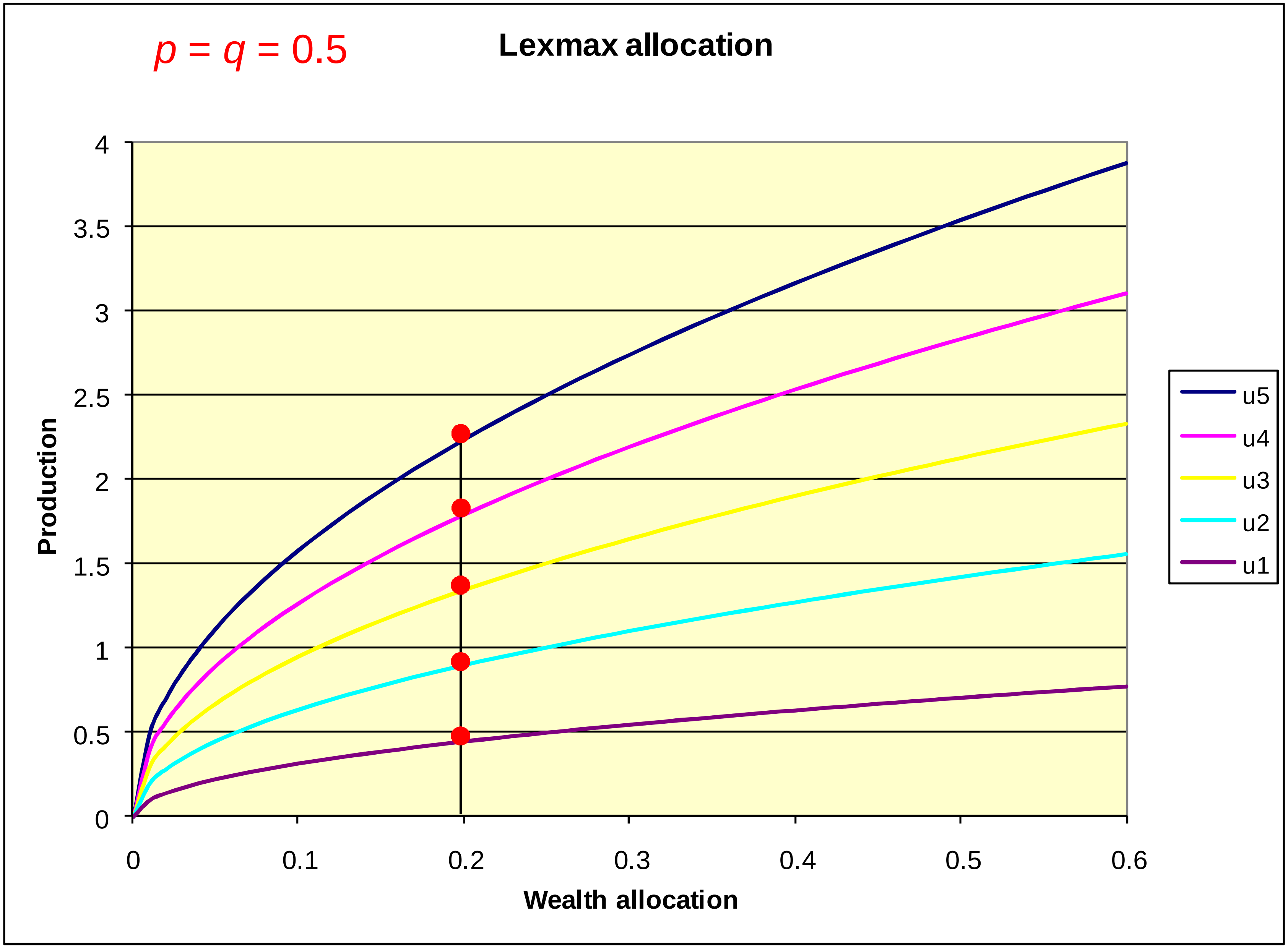
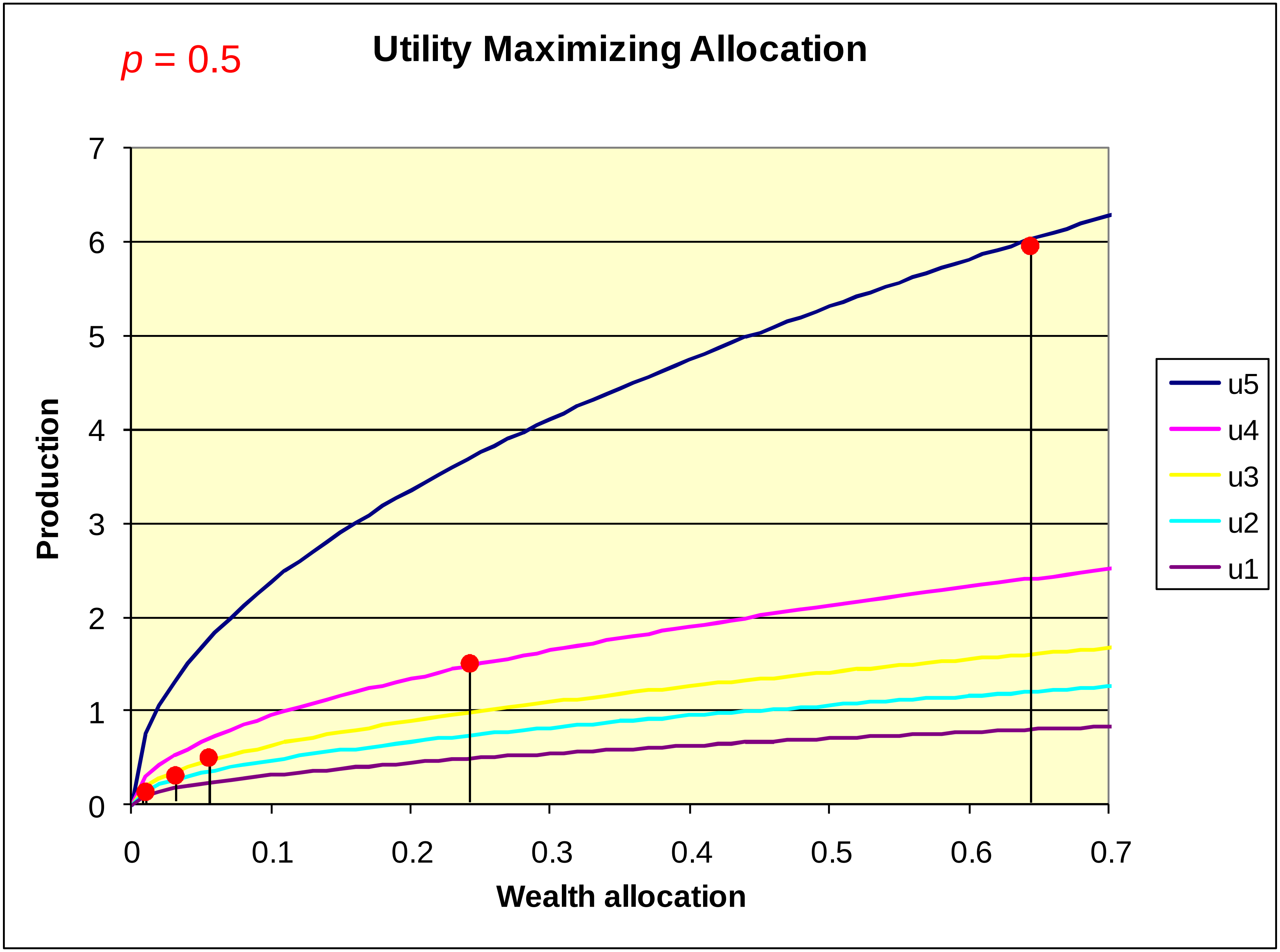
Model now **simplifies**.



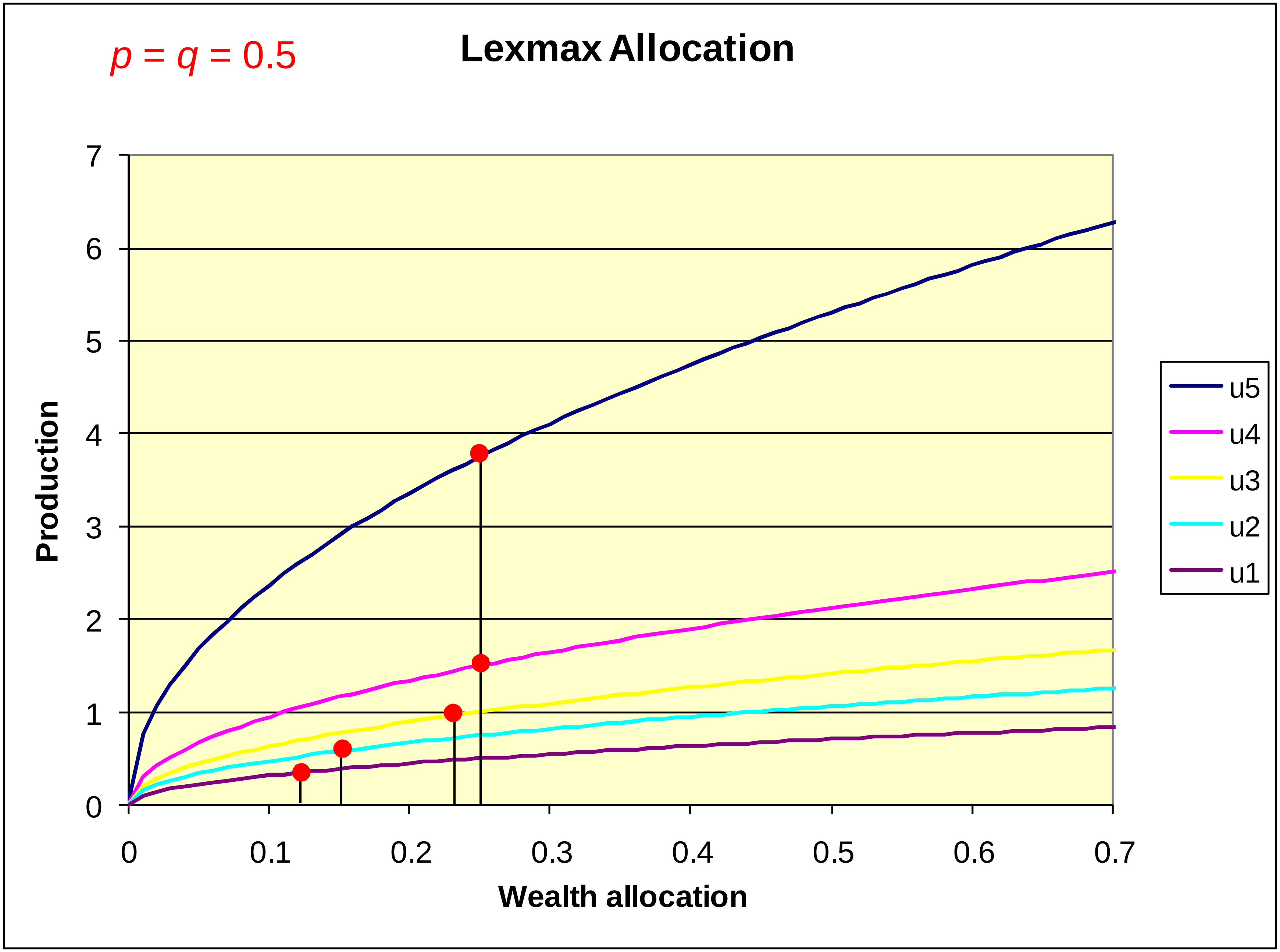
p = 0.5
Utility maximizing allocation
β = 0
Production
Wealth allocation
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
u5
u4
u3
u2
u1



p = q = 0.5
Lexmax allocation
Production
Wealth allocation
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
u5
u4
u3
u2
u1



p = 0.5
Utility Maximizing Allocation
Production
7
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Wealth allocation
u5
u4
u3
u2
u1



$p = q = 0.5$
Lexmax Allocation
Production
7
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Wealth allocation
u5
u4
u3
u2
u1

Lexmax Model

- When does the Rawlsian model result in equality?
  - That is, when do we have $x_1 = \cdots = x_n$ in the solution of the lexmax problem?

Lexmax Model

- KKT conditions for equality :

$$
\begin{aligned}
&2\mu_1 - \mu_2 = d_1 \\
&\mu_1 + \mu_i - \mu_{i+1} = d_i, \quad i = 2,\ldots,n-2 \\
&\mu_1 + \mu_{n-1} = d_{n-1}
\end{aligned}
$$

- with RHS's:

$$
d_i = v(x_i)\frac{\sum_i c_i u_i(x_i)}{\sum_i v(x_i)}\left(\frac{v'(x_1)}{v(x_1)} - \frac{u_{i+1}{}'(x_{i+1}) - u_1{}'(x_1)}{\sum_i c_i u_i(x_i)} + \frac{v'(x_{i+1}) - v'(x_1)}{\sum_i v(x_i)}\right)
$$

- Remarkably, these can be solved in closed form, yielding…

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \leq \frac{q}{p} \cdot \frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n-1$.

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\underbrace{\boxed{\frac{1}{n-k}\sum_{i=k+1}^{n} c_i}}_{\text{Average of } n-k \text{ largest } c_i\text{'s}} - \underbrace{\boxed{\frac{1}{k}\sum_{i=1}^{k} c_i}}_{\text{Average of } k \text{ smallest } c_i\text{'s}} \le \frac{q}{p} \cdot \frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n - 1$.

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \le \frac{q}{\boxed{p}} \cdot \frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n-1$.

- Equality is **more likely** to be required when p is small.
  - When investment in an individual yields rapidly decreasing marginal returns.

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \leq \frac{q}{p} \cdot \frac{n-\boxed{k}}{\boxed{k}}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n-1$.

- Equality test is **more sensitive** at upper end (large k).
  - Equality is **unlikely** to be required when the productivity distribution has a long **upper** tail.
  - Equality **may be required** even when the distribution has a long **lower** tail.

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \le \frac{\boxed{q}}{p} \cdot \frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n-1$.

- Equality is **more likely** to be required when q is large.
  - When people **want to get rich**.

Utilitarianism + Equity

Joint work with
H. P. Williams, London School of Economics

A Triage Model

- Utilitarian and Rawlsian distributions seem **too extreme** in practice.
  - How to combine them?

A Triage Model

- Utilitarian and Rawlsian distributions seem **too extreme** in practice.
  - How to combine them?
- Focus on **health care**
  - Allocation of resources
- **Triage** model
  - Maximize welfare of **most seriously ill** (Rawlsian)...
  - …until this requires **undue sacrifice** from others

A Triage Model

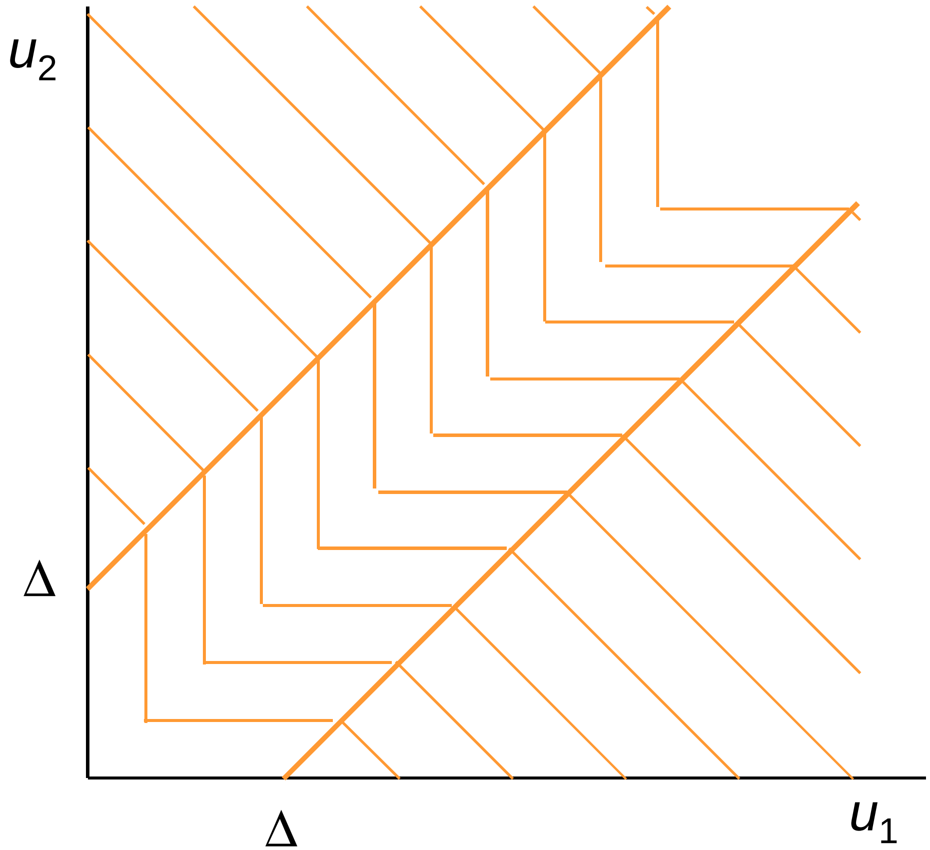
- Proposal:
  - Switch from **Rawlsian** to **utilitarian** when **inequality** exceeds Δ.

A Triage Model

- Proposal:
  - Switch from **Rawlsian** to **utilitarian** when **inequality** exceeds Δ.
  - Let u_i = utility allocated to person i
- For 2 persons:
  - Maximize $\min_i \{u_1, u_2\}$ (Rawlsian) when $|u_1 - u_2| \leq \Delta$
  - Maximize $u_1 + u_2$ (utilitarian) when $|u_1 - u_2| > \Delta$

A Triage Model

Contours of **social welfare function** for 2 persons.



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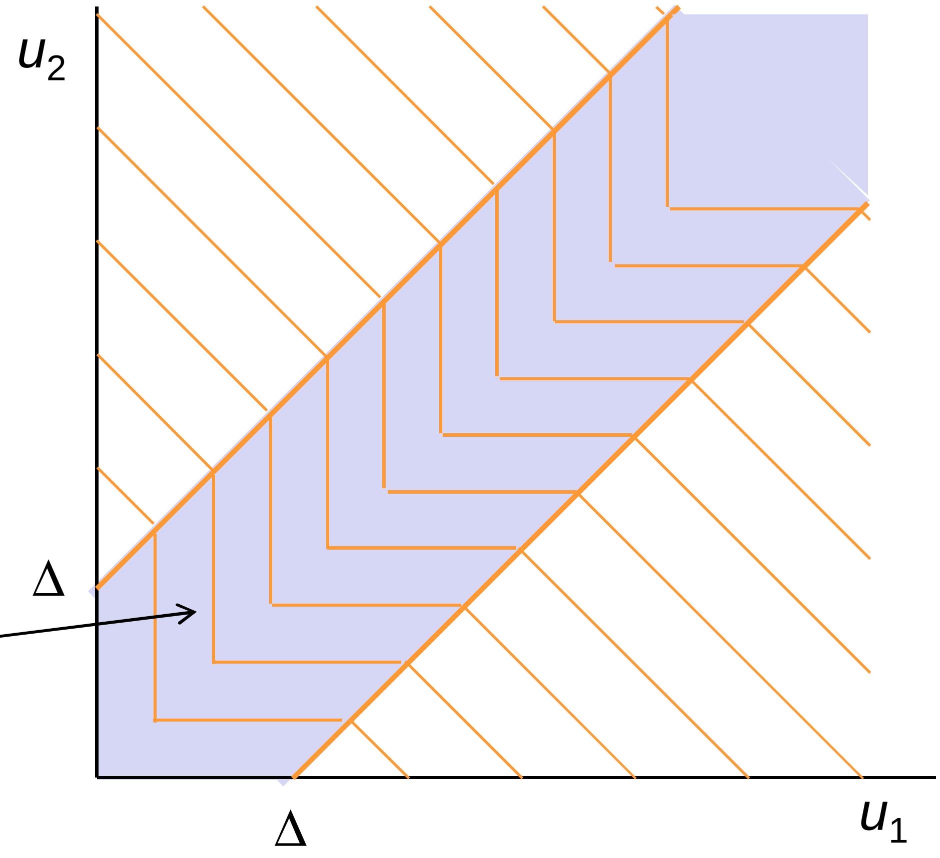


Rawlsian region

$\min\{u_1, u_2\}$

A Triage Model

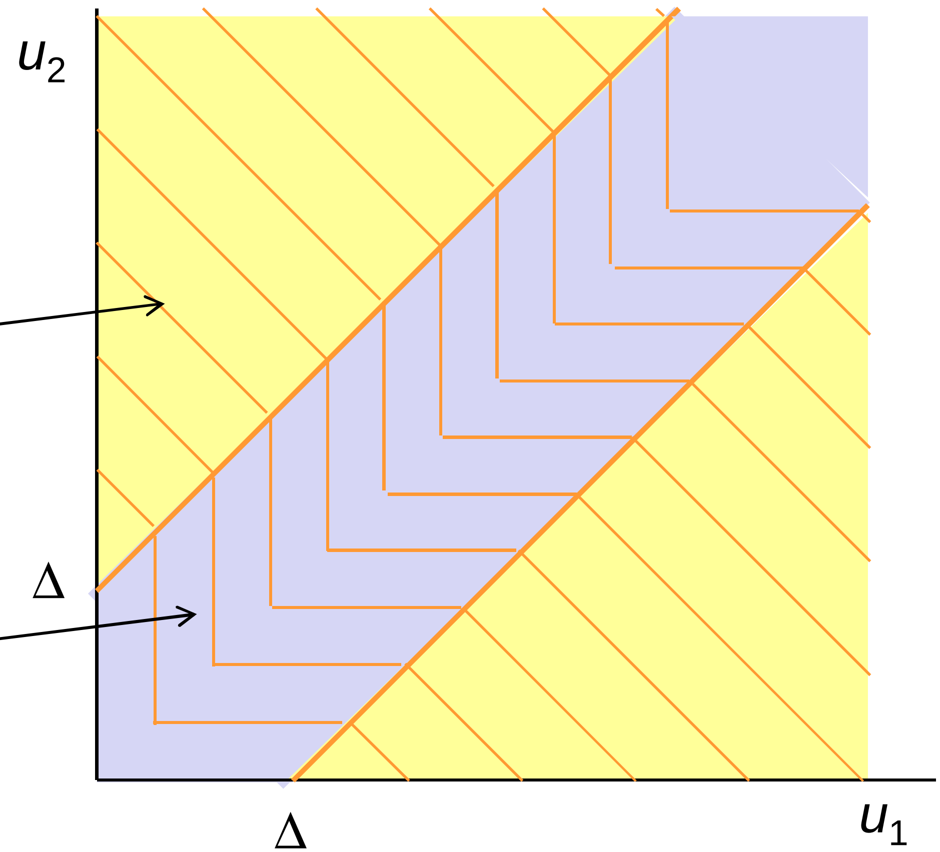
Contours of **social welfare function** for 2 persons.

Utilitarian region

$u_1 + u_2$

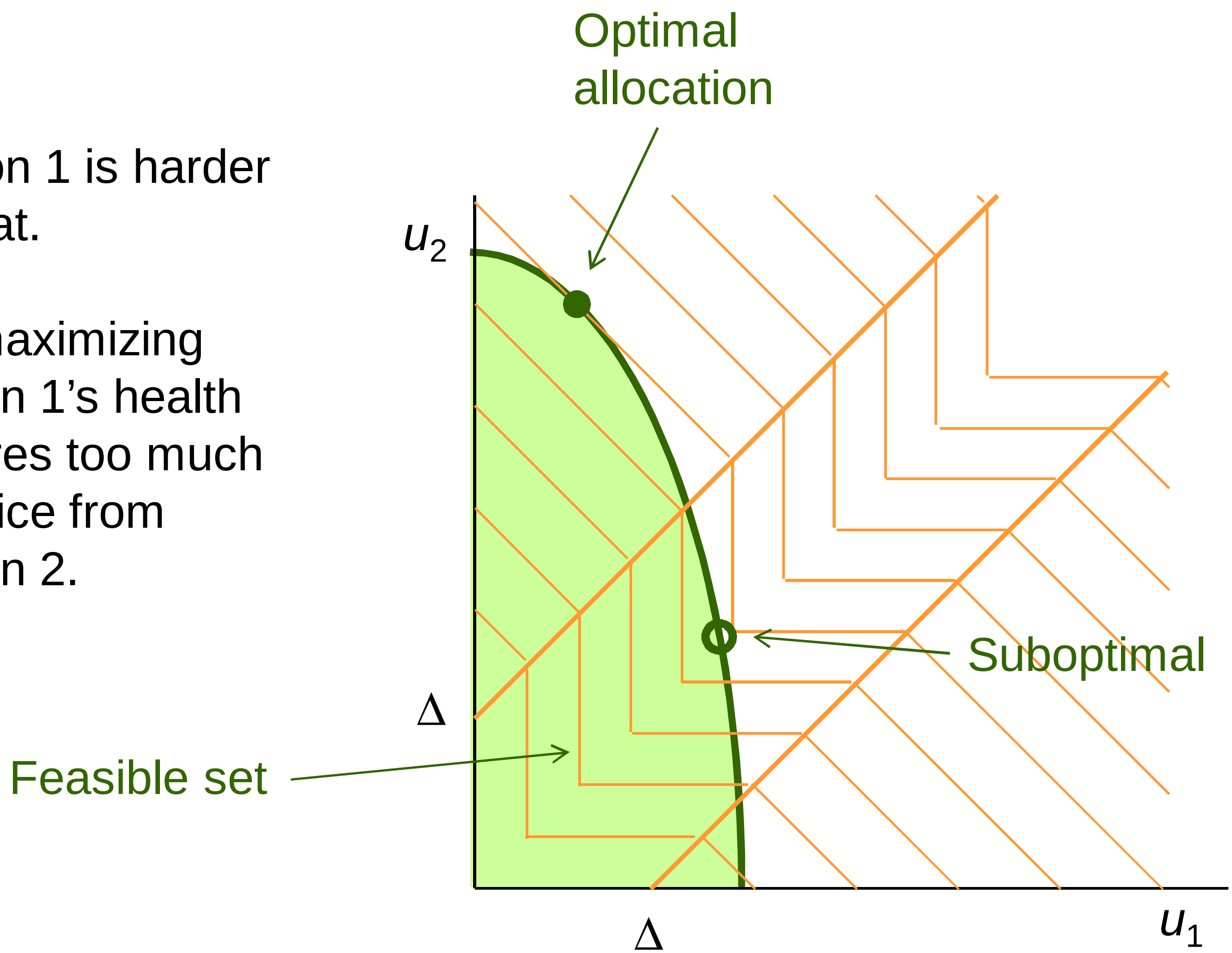
Rawlsian region

$\min\{u_1, u_2\}$



Person 1 is harder to treat.

But maximizing person 1's health requires too much sacrifice from person 2.

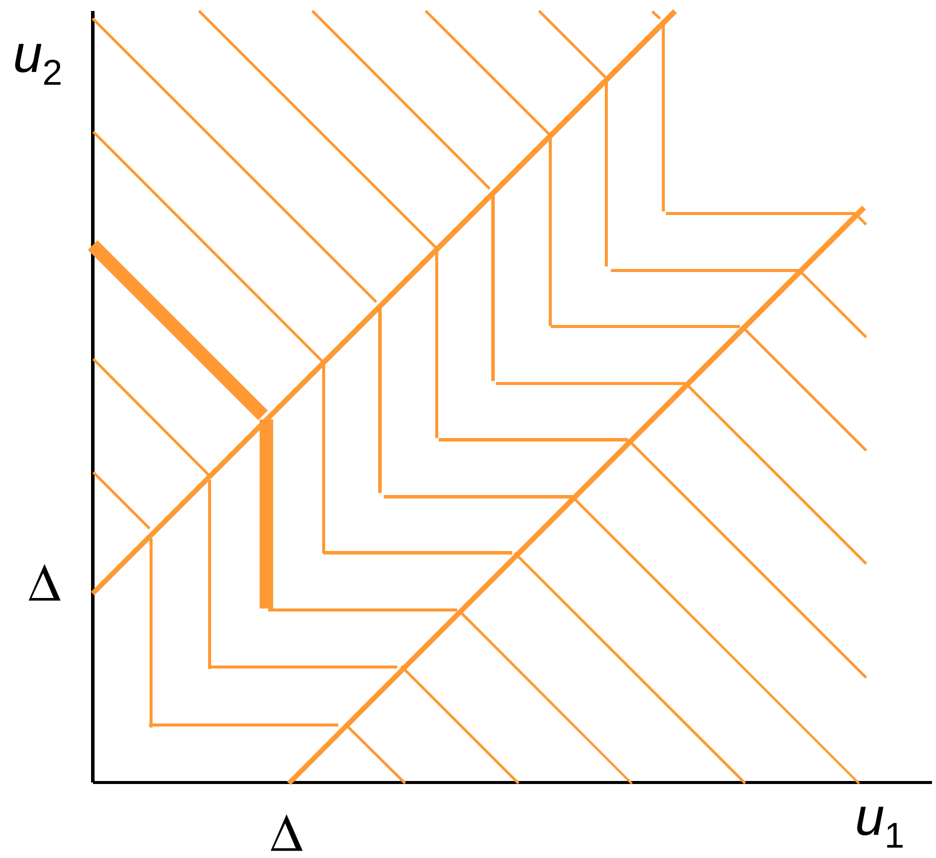


A Triage Model

- Advantage: Only **one parameter Δ**
  - **Focus** for debate.
  - Δ has **intuitive meaning** (unlike weights)
  - Examine **consequences** of different settings for Δ
  - Find **least objectionable** setting
  - Results in a **consistent** policy

A Triage Model

We want continuous contours…



A Triage Model

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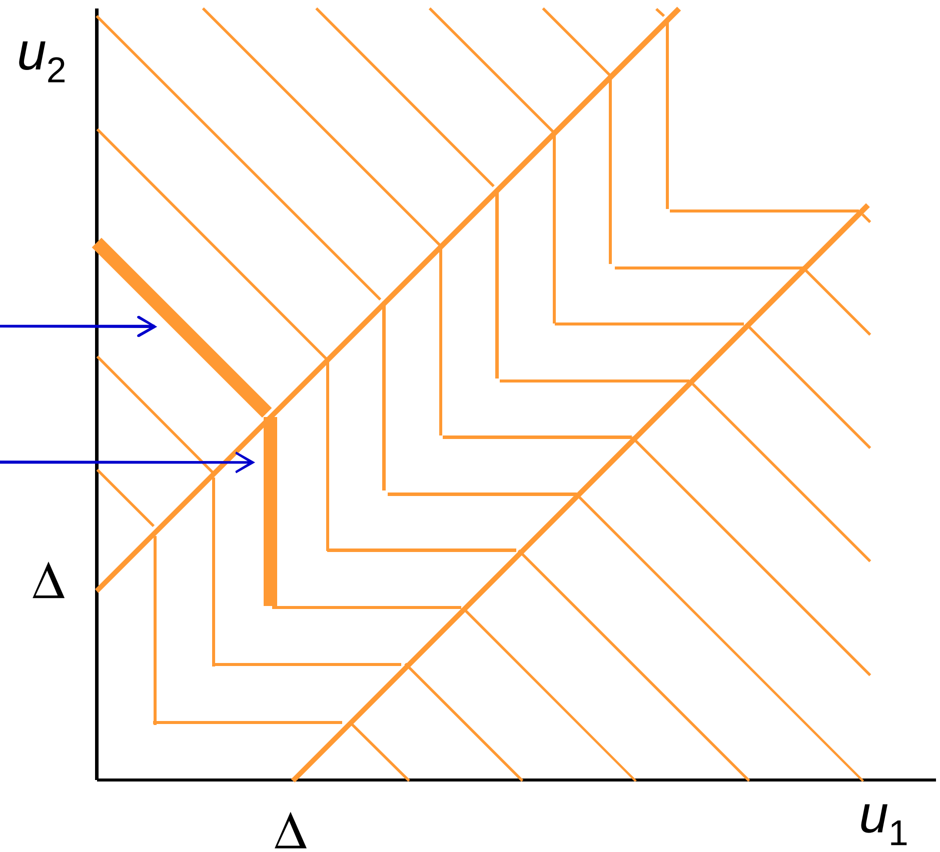


So we use affine transform of Rawlsian criterion

A Triage Model

The social welfare problem becomes

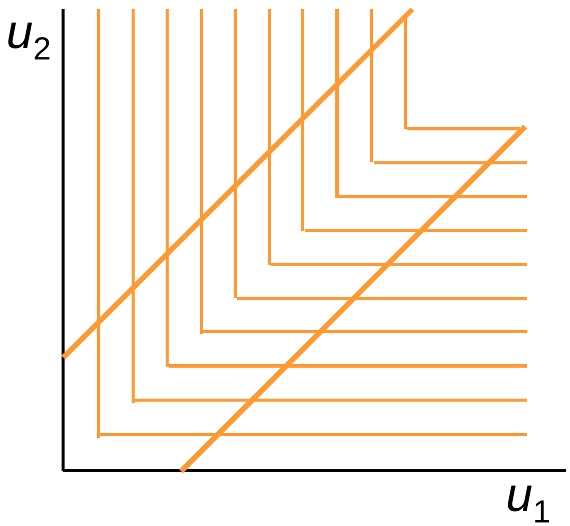
$$\max \ z$$

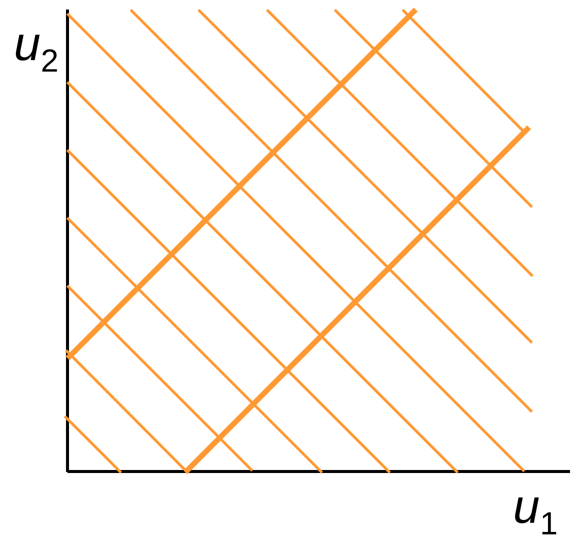
$$z \le \begin{cases} 2\min\{u_1, u_2\} + \Delta, & \text{if } |u_1 - u_2| \le \Delta \\ u_1 + u_2, & \text{otherwise} \end{cases}$$

constraints on feasible set

MILP Model

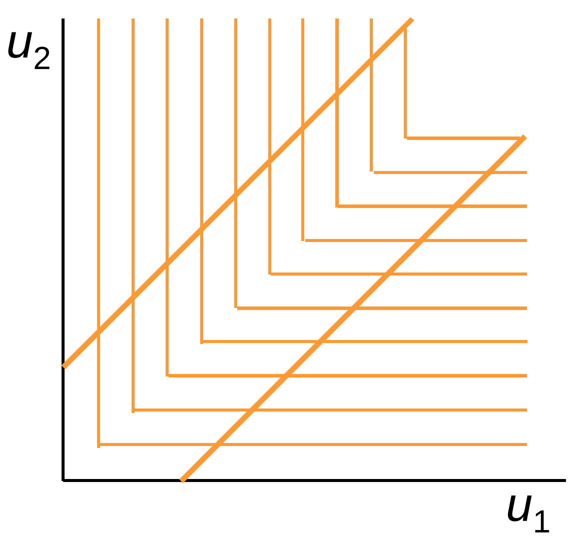
Epigraph is union of 2 polyhedra.

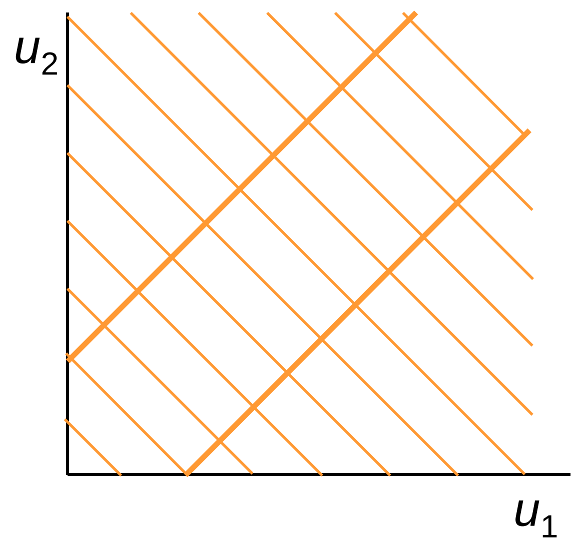




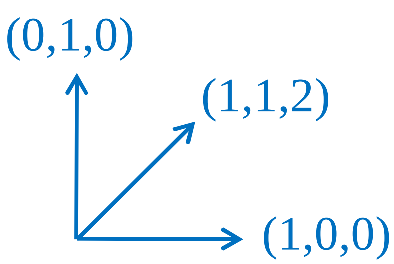
MILP Model

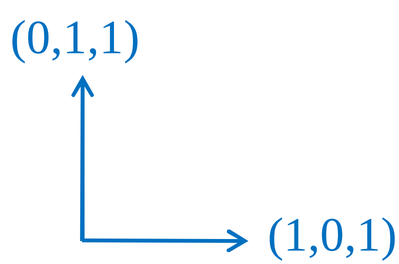
Epigraph is union of 2 polyhedra.
Because they have **different recession cones**, there is no MILP model.





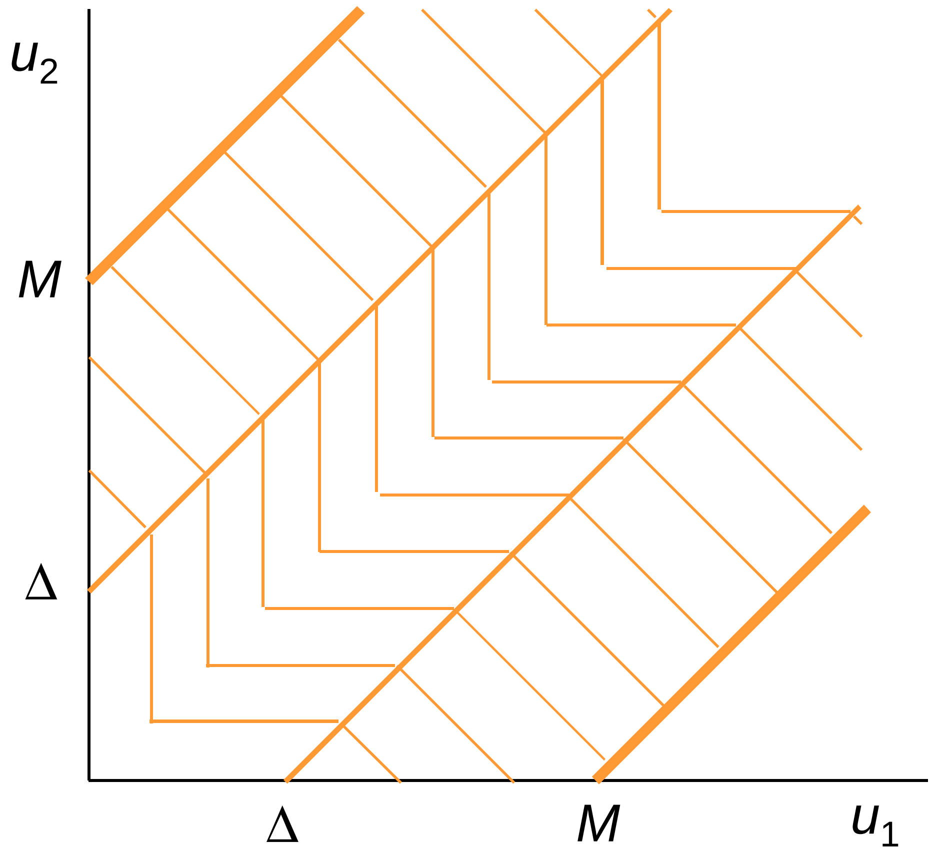
Recession directions (u_1,u_2,z)





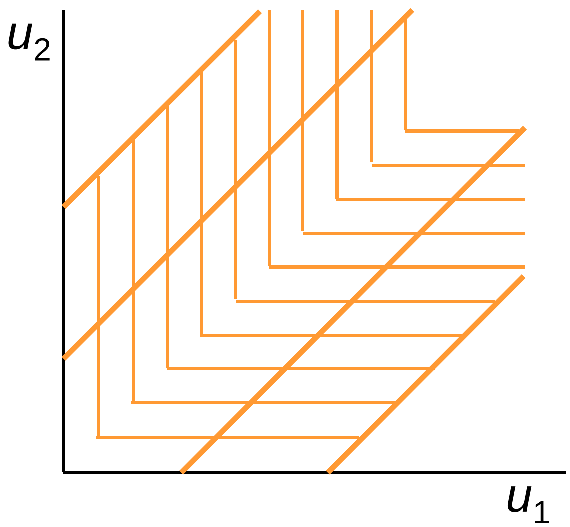
MILP Model

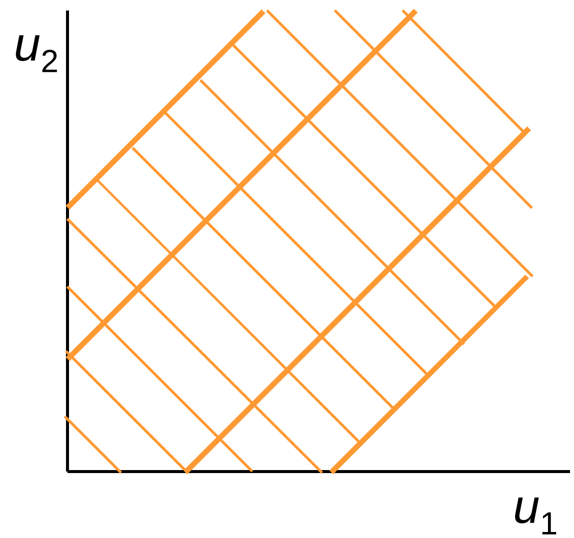
Impose constraints $|u_1 - u_2| \leq M$



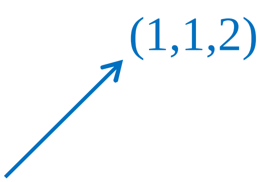
MILP Model

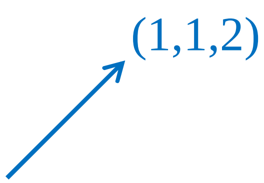
This equalizes recession cones.





Recession directions (u_1,u_2,z)





MILP Model

We have the model…

$$\max\ z$$

$$z \le 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1,2$$

$$z \le u_1 + u_2 + \Delta(1 - \delta)$$

$$u_1 - u_2 \le M, \quad u_2 - u_1 \le M$$

$$u_1, u_2 \ge 0$$

$$\delta \in \{0,1\}$$

constraints on feasible set

u_1

MILP Model

We have the model…

$$
\begin{aligned}
&\max\ z \\
&z \le 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1,2 \\
&z \le u_1 + u_2 + \Delta(1 - \delta) \\
&u_1 - u_2 \le M, \quad u_2 - u_1 \le M \\
&u_1, u_2 \ge 0 \\
&\delta \in \{0,1\}
\end{aligned}
$$

u_1

This is a **convex hull** formulation.

n-person Model

Rewrite the 2-person social welfare function as…

$$\Delta + 2u_{\min} + \left(u_1 - u_{\min} - \Delta\right)^+ + \left(u_2 - u_{\min} - \Delta\right)^+$$

$\min\{u_1, u_2\}$

$\alpha^+ = \max\{0, \alpha\}$

u_1

n-person Model

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$\min\{u_1, u_2\}$

$\alpha^+ = \max\{0, \alpha\}$

This can be generalized to *n* persons:

$$(n-1)\Delta + nu_{\min} + \sum_{j=1}^{n}\left(u_j - u_{\min} - \Delta\right)^+$$

u_1

n-person Model

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This can be generalized to *n* persons:

$$(n-1)\Delta + nu_{\min} + \sum_{j=1}^{n}\left(u_j - u_{\min} - \Delta\right)^+$$

u_1

Epigraph is a union of $n!$ polyhedra with same recession direction $(u,z) = (1,\ldots,1,n)$ if we require $|u_i - u_j| \le M$

So there is an MILP model…

n-person MILP Model

To avoid $n!$ 0-1 variables, add auxiliary variables w_{ij}

$$\begin{aligned}
&\max\ z \\
&z \le u_i + \sum_{j \ne i} w_{ij},\ \text{all } i \\
&w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta),\ \text{all } i, j \text{ with } i \ne j \\
&w_{ij} \le u_j + (1 - \delta_{ij})\Delta,\ \text{all } i, j \text{ with } i \ne j \\
&u_i - u_j \le M,\ \text{all } i, j \\
&u_i \ge 0,\ \text{all } i \\
&\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j
\end{aligned}$$

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&u_i \ge 0,\ \text{all } i \\
&\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j
\end{aligned}
$$

u_1

Theorem. The model is correct (not easy to prove).

n-person MILP Model

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$$

u_1

Theorem. The model is correct (not easy to prove).

Theorem. This is a convex hull formulation (not easy to prove).

Health Example

Measure utility in QALYs (quality-adjusted life years).
Assume all groups have equal size.

Intervention	**Cost per QALY** c_i	**QALYs without intervention** α_j	**Demand** L_j
Pacemaker	£700	5	£7000
Hip replacement	£750	15	£7500
Coronary bypass	£2400	3	£24,000
Heart transplant	£5000	1	£50,000
Kidney dialysis	£11,000	1	£100,000

From literature

Hypothetical (depends on target group)

Health Example

Add constraints to define feasible set…

$$\max\ z$$

$$z \le u_i + \sum_{j \ne i} w_{ij},\ \text{all } i$$

$$w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta),\ \text{all } i, j \text{ with } i \ne j$$

$$w_{ij} \le u_j + (1 - \delta_{ij})\Delta,\ \text{all } i, j \text{ with } i \ne j$$

$$u_i - u_j \le M,\ \text{all } i, j$$

$$u_i \ge 0,\ \text{all } i$$

u_1

$$\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j$$

$$\boxed{\begin{array}{l} u_i = \alpha_i + c_i x_i,\quad \text{all } i \\ \sum_i x_i \le \text{budget} \\ 0 \le x_i \le L_i,\quad \text{all } i \end{array}}$$

Health Example

Assume budget of £100,000.
Optimal allocations (£) for various Δ (boldface if at upper bound):

Intervention	Δ = 0	Δ = 5	Δ = 10	Δ = ∞
Pacemaker	**7000**	**7000**	869	1143
Hip replacement	**7500**	**7500**	**7500**	0
Coronary bypass	**24,000**	6978	7778	8720
Heart transplant	**50,000**	24,538	26,204	28,168
Kidney dialysis	11,500	53,984	57,649	61,969

Utilitarian

Rawlsian

Generalization

- Target groups may have **different sizes**.

  $\max\{u_{min}, f\}$

  - Lexmax and inequality threshold Δ apply to **individuals**.
  - Utility applies to **groups.**

- The model and theorems **generalize** to this case.

Future Work

- Survey justice/equity/equality criteria and investigate their **properties as optimization problems**.
  - Range, range ratio, relative mean deviation
  - McLoone index
  - Variance, (logarithmic) coefficient of variation
  - Gini coefficient
  - Atkinson's measure
  - Hoover index
  - Theil index (entropy)
  - Nash bargaining
  - Raiffa-Kalai-Smorodinsky bargaining
  - Triage