Optimality Conditions for Distributive Justice

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Just Distribution

- **The problem:** How to distribute resources…

Salaries

Tax breaks

Medical care

Disaster relief

Government benefits

Justice and Optimization

- The problem is not to satisfy preferences, but to **achieve justice.**

Justice and Optimization

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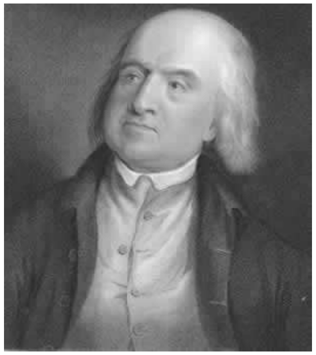
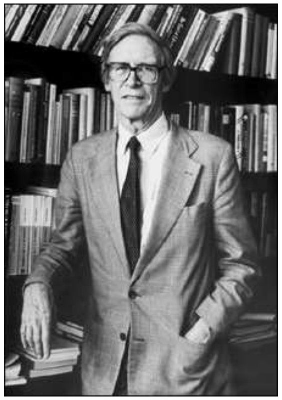
- Two classical criteria for distributive justice:
  - **Utilitarianism**
  - **Difference principle** of **John Rawls**





Justice and Optimization

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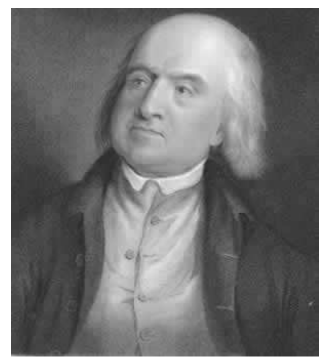

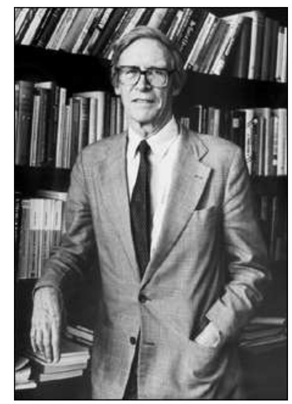

- Both can be viewed as **mathematical optimization problems.**

Justice and Optimization

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- The **Rawlsian difference principle** calls for a lexicographic maximum of utilities allotted to individuals**.**

- **Optimization theory** can…
  - provide some **insight** into when a distribution of wealth is just.
  - Allow us to **calculate** just allocations of resources.

Justice and Optimization

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  - Unlike most mathematical/axiomatic treatments of social welfare.

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- Distribution of greater resources to **more productive individuals** may increase overall utility.
  - i.e., to individuals who are **more talented**, **better positioned socially**, or **work harder**.

Justice and Optimization

- We analyze distributions over **nonidentical individuals.**
  - Unlike most mathematical/axiomatic treatments of social welfare.
- Distribution of greater resources to **more productive individuals** may increase overall utility.
  - i.e., to individuals who are **more talented**, **better positioned socially**, or **work harder**.
- To what extent does **efficiency require inequality** in the utilitarian and Rawlsian models?

Outline

- Utilitarian model
- Utilitarian model with cost of social disharmony
- Rawlsian model

Utilitarian Model

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  $u_i(x_i)$ = utility eventually produced by person *i*

Utilitarian Model

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- Let x_i = wealth initially allocated to person *i*
  $u_i(x_i)$ = utility eventually produced by person *i*

- Conceptually, $u_i(x_i) = v(o(x_i))$

Utility as a function of wealth

Wealth creation as a function of wealth input

Utilitarian Model

- The utility maximization problem:

$$\max \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1 \quad \text{Total budget}$$

$$x_i \geq 0, \quad \text{all } i$$

Utilitarian Model

- To solve it:

$$\max \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0, \quad \text{all } i$$

Associate Lagrange multiplier λ with this constraint

Utilitarian Model

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$$\sum_{i=1}^{n} x_i = 1$$

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Associate Lagrange multiplier λ with this constraint

- Any solution in which each $x_i \geq 0$ satisfies

$$\frac{\partial}{\partial x_i} L(x,\lambda) = \frac{\partial}{\partial x_i}\left(\sum_i u_i(x_i) - \lambda \sum_i x_i\right) = u_i'(x_i) - \lambda = 0, \quad \text{all } i$$

Utilitarian Model

- So $u_1'(x_1) = \cdots = u_n'(x_n)$

Marginal productivity

Distribute wealth so as to equalize marginal productivity.

Utilitarian Model

- So $u_1'(x_1) = \cdots = u_n'(x_n)$

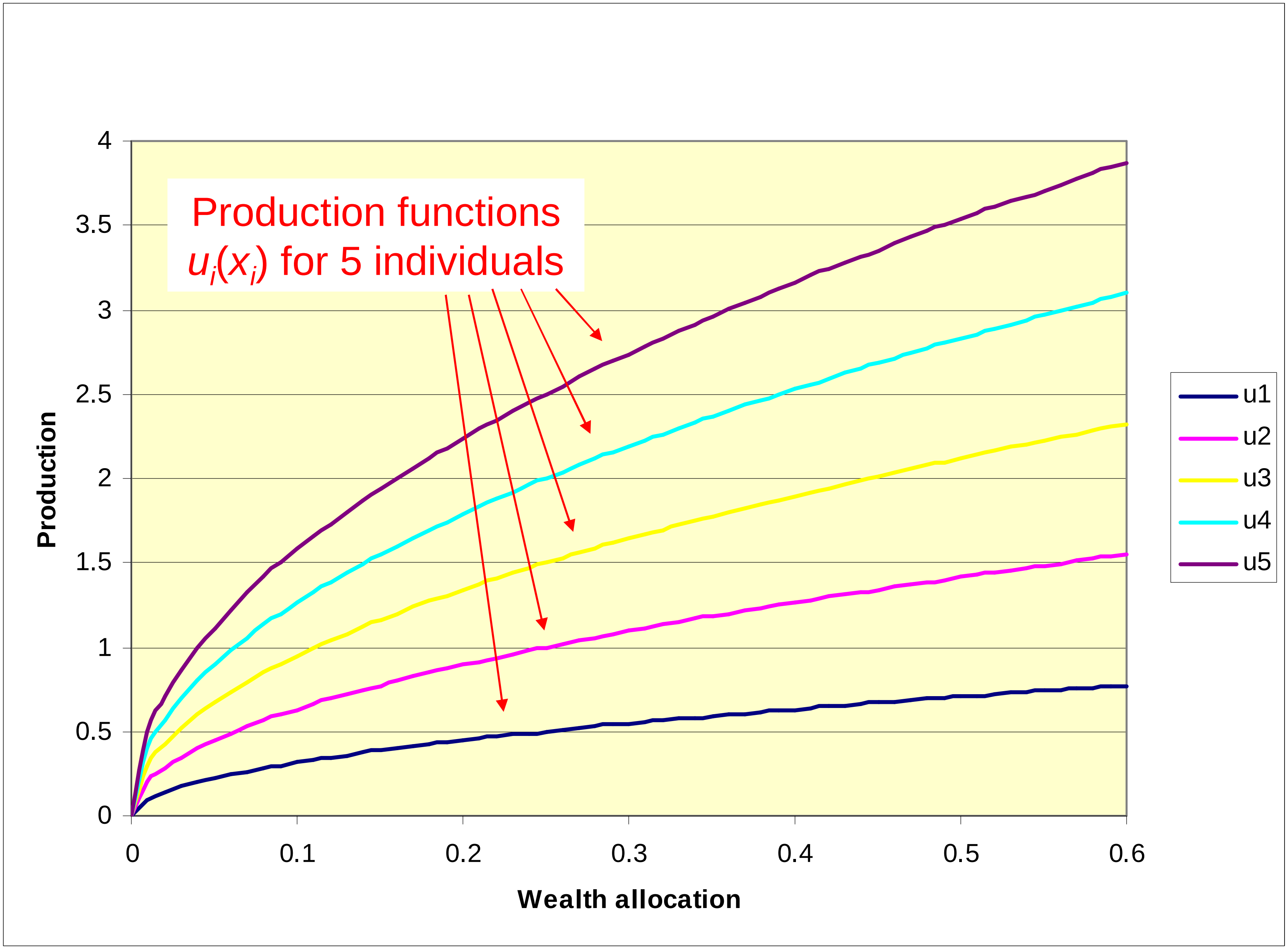
  Marginal productivity

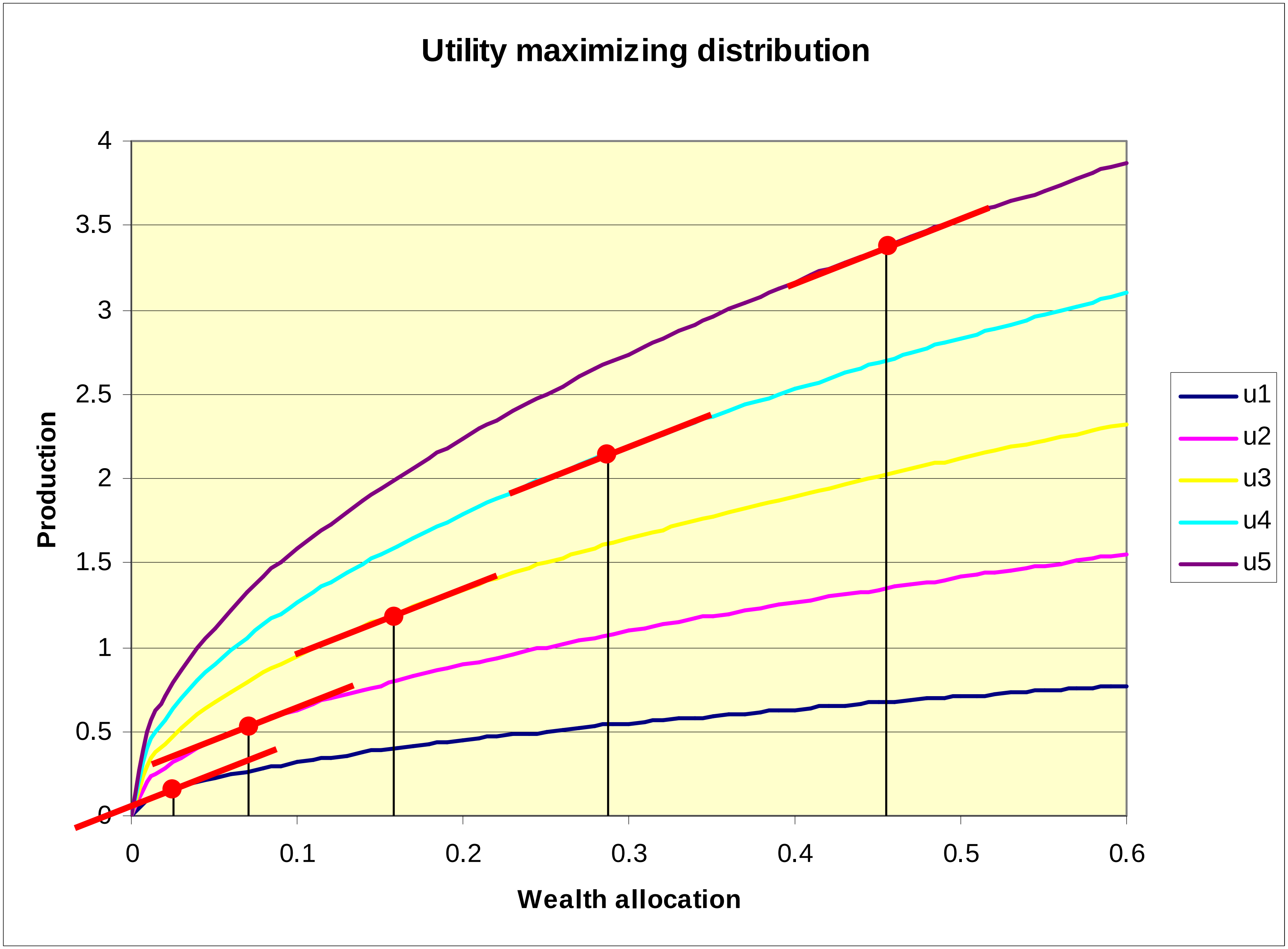
  Distribute wealth so as to equalize marginal productivity.

- If we assume persons are indexed in order of marginal productivity, i.e., $u_i'(\cdot) \le u_{i+1}'(\cdot)$, all i

  Then $x_1 \le \cdots \le x_n$

  Less productive individuals receive less wealth.


Production functions
$u_i(x_i)$ for 5 individuals
Production
Wealth allocation
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u1
u2
u3
u4
u5


Utility maximizing distribution
Production
Wealth allocation
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u1
u2
u3
u4
u5

Utilitarian Model

- An **egalitarian distribution** $x_1 = \cdots = x_n$ is optimal only when

$$u_1'(1/n) = \cdots = u_n'(1/n)$$

- So, equality is optimal only when everyone has the same marginal productivity in an egalitarian distribution.

Utilitarian Model

- Let $u_i(x_i) = c_i x_i^p$ where $p \geq 0$
- Then the optimal wealth distribution is

$$x_i = c_i^{\frac{1}{1-p}} \left(\sum_{j=1}^{n} c_j^{\frac{1}{1-p}} \right)^{-1}$$

Utilitarian Model

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- When $p \geq 1$: $x_n = 1$ and all other $x_i = 0$.
  - The most productive individual gets **everything**.

Utilitarian Model

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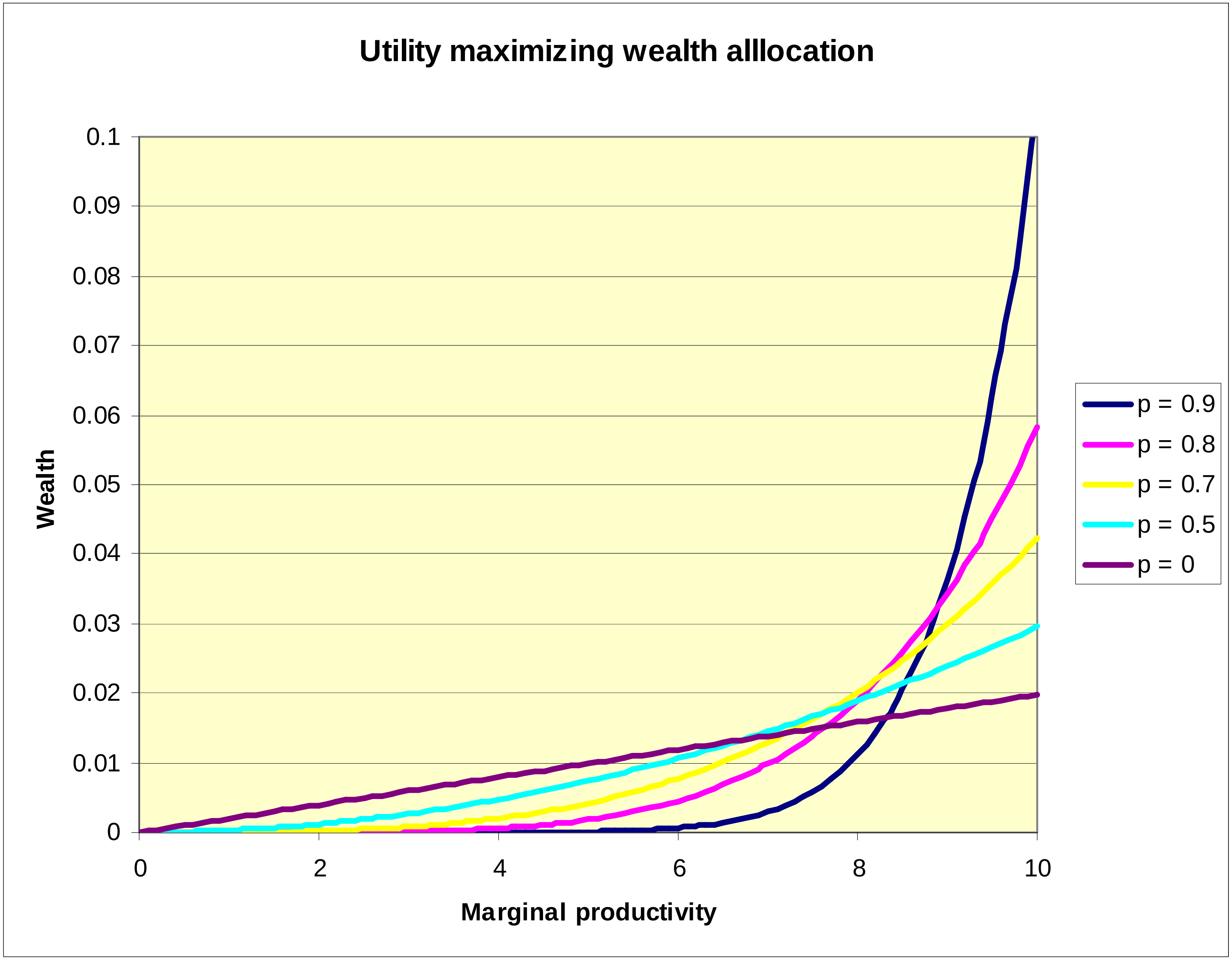
- When $p \geq 1$: $x_n = 1$ and all other $x_i = 0$.
  - The most productive individual gets **everything**.
- When $p < 1$:
  - Distribution is **egalitarian** only if $c_1 = \cdots = c_n$
  - Otherwise the **most egalitarian** distribution occurs when $p = 0$: $x_i = \dfrac{c_i}{\sum_j c_j}$

Utilitarian Model

- So if productivity is at least proportional to input ($p \geq 1$), the most productive class gets everything.

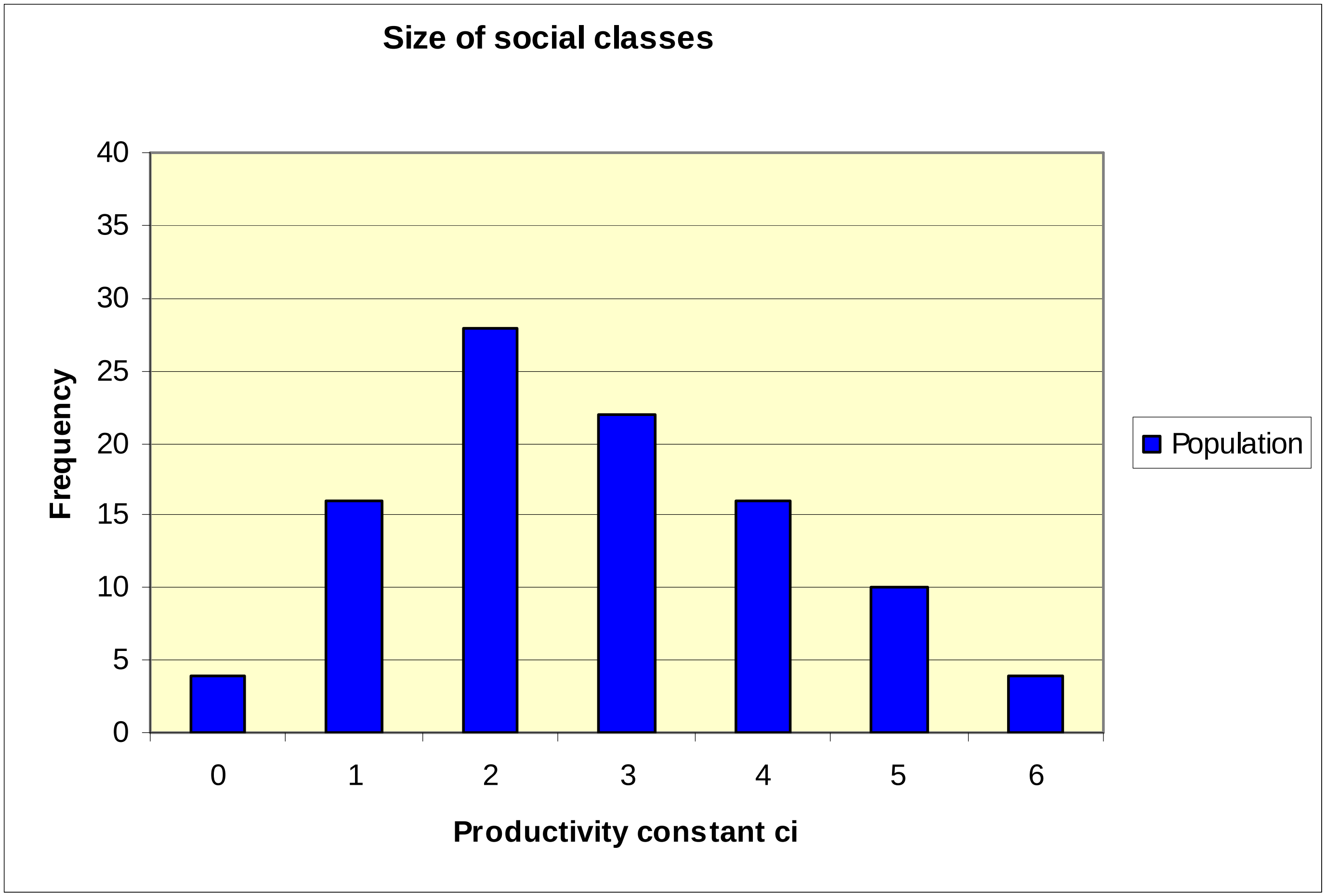
- Otherwise, the most nearly egalitarian distribution that can be optimal is one in which people receive wealth in proportion to c_i.
  - And this occurs only when productivity is not sensitive to investment ($p = 0$).

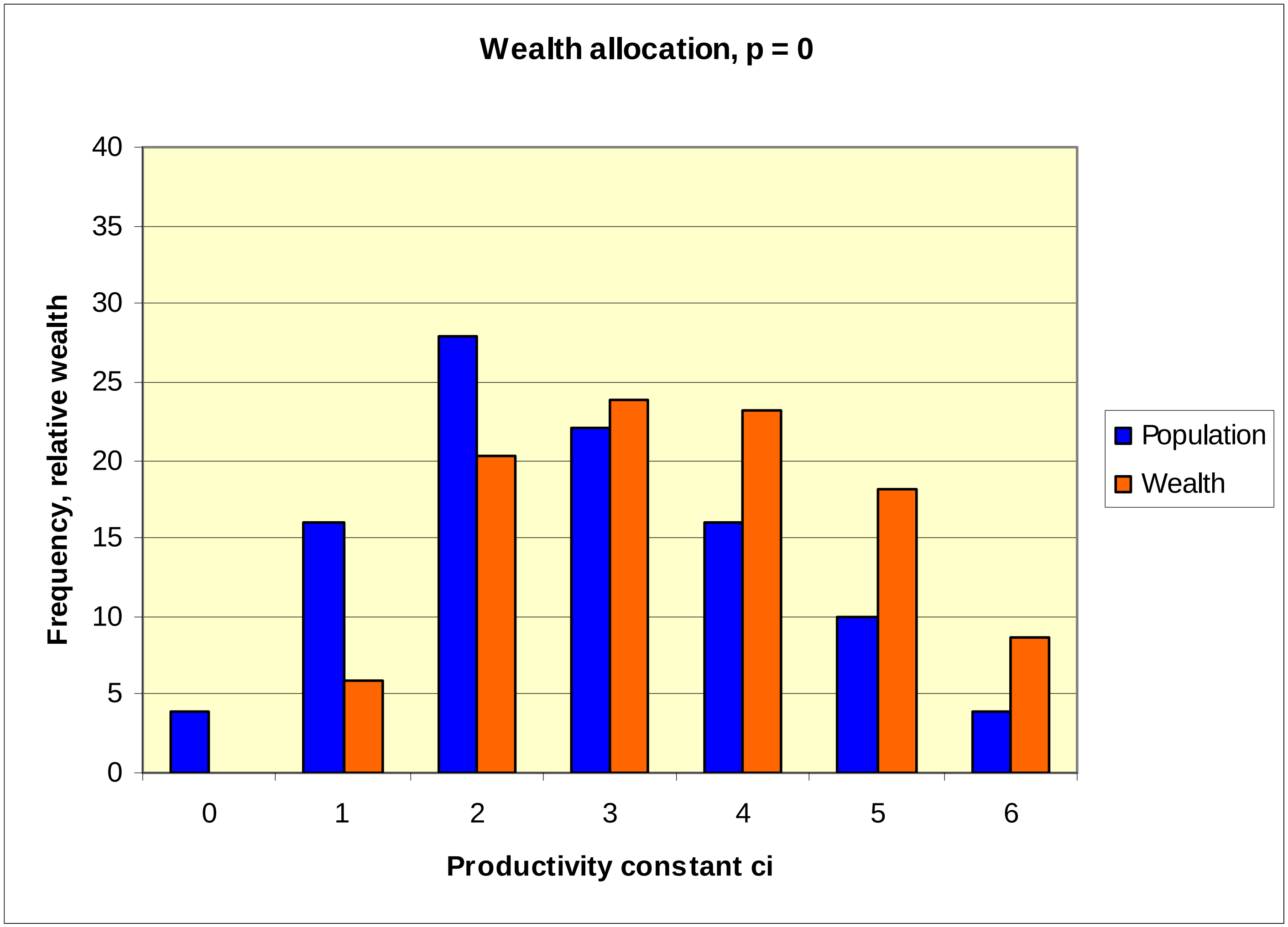


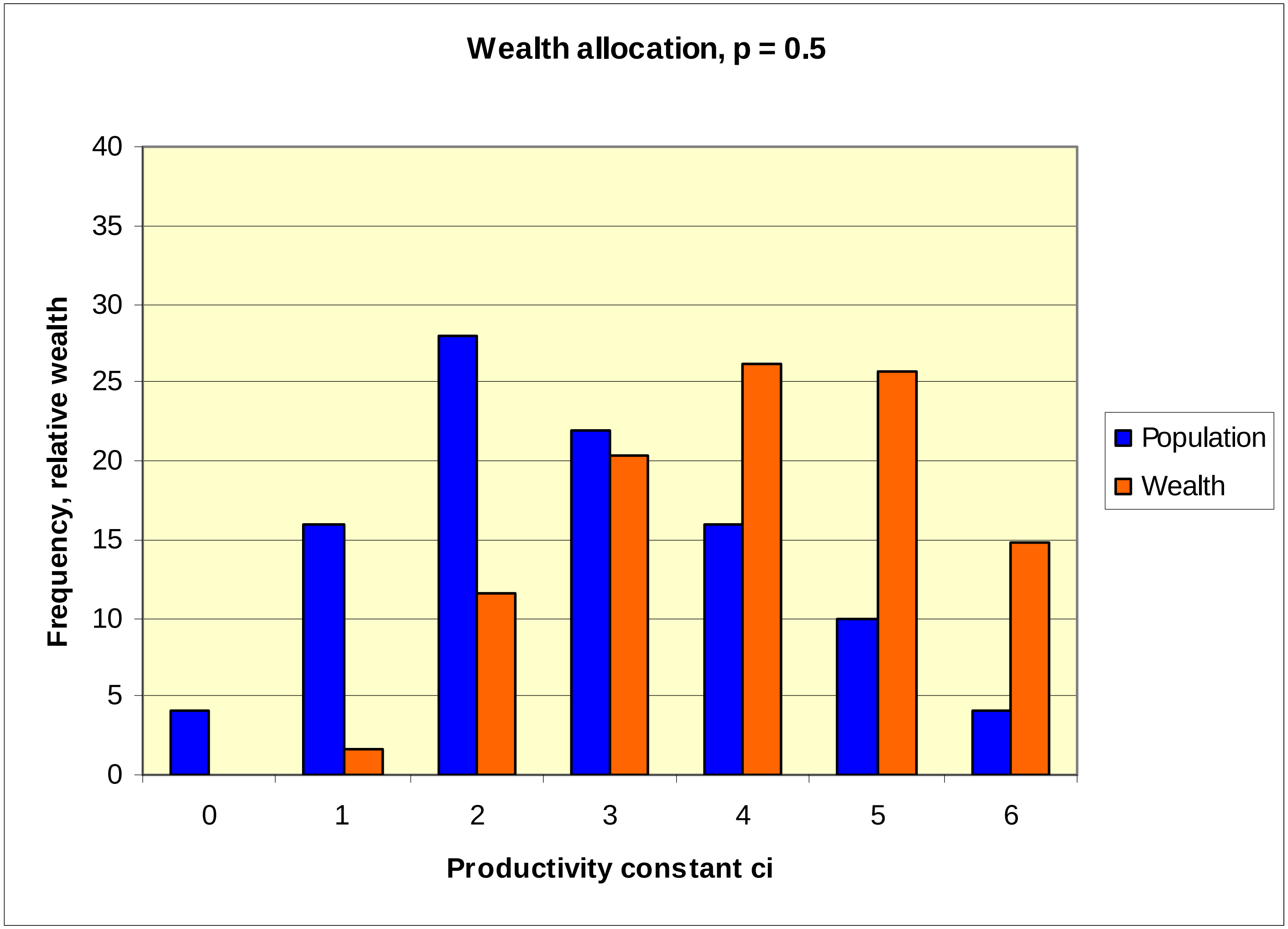
Utility maximizing wealth alllocation
Wealth
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
2
4
6
8
10
Marginal productivity
p = 0.9
p = 0.8
p = 0.7
p = 0.5
p = 0

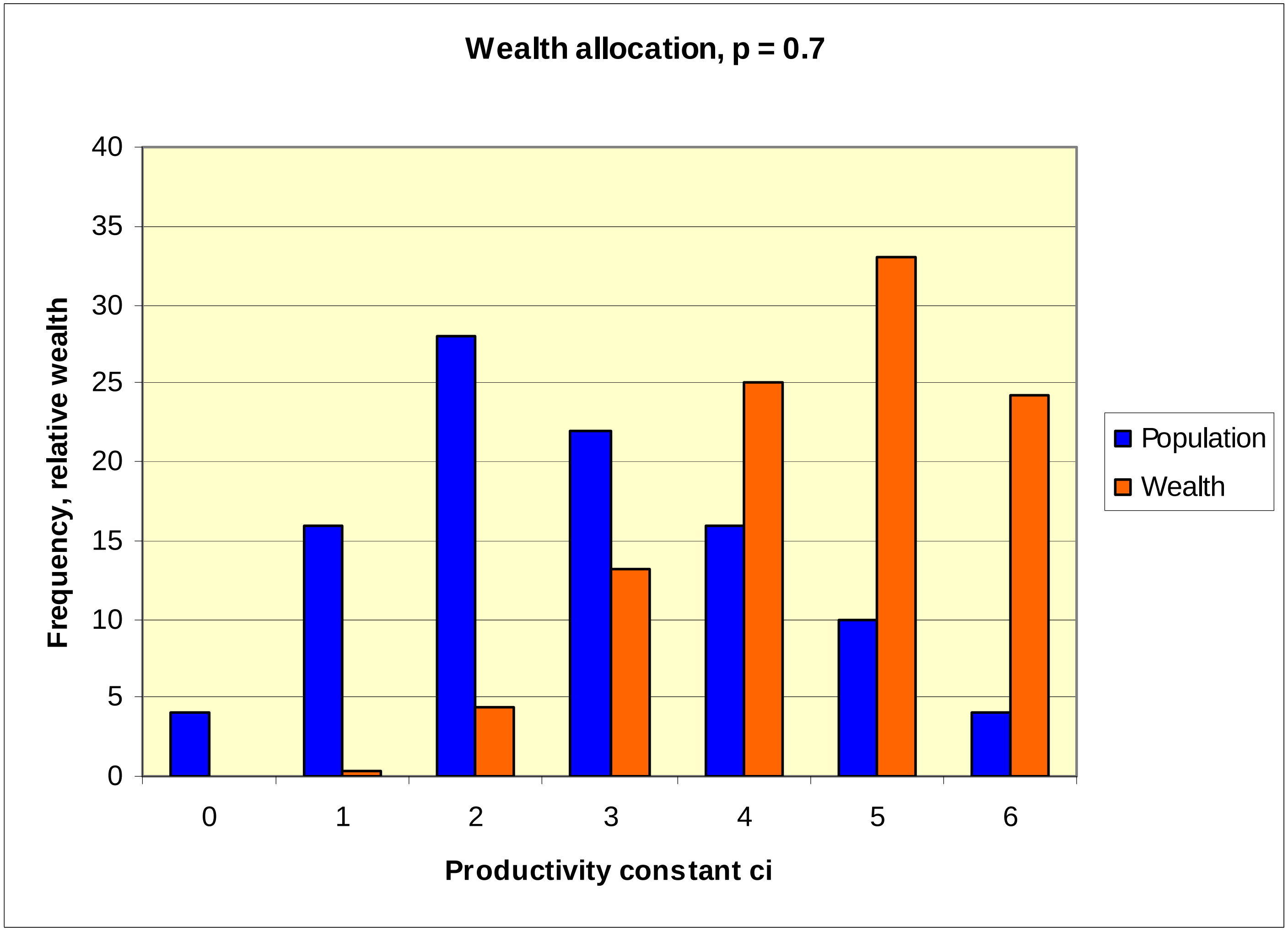
Utilitarian Model

- Let's see how wealth is distributed in a multiclass society…

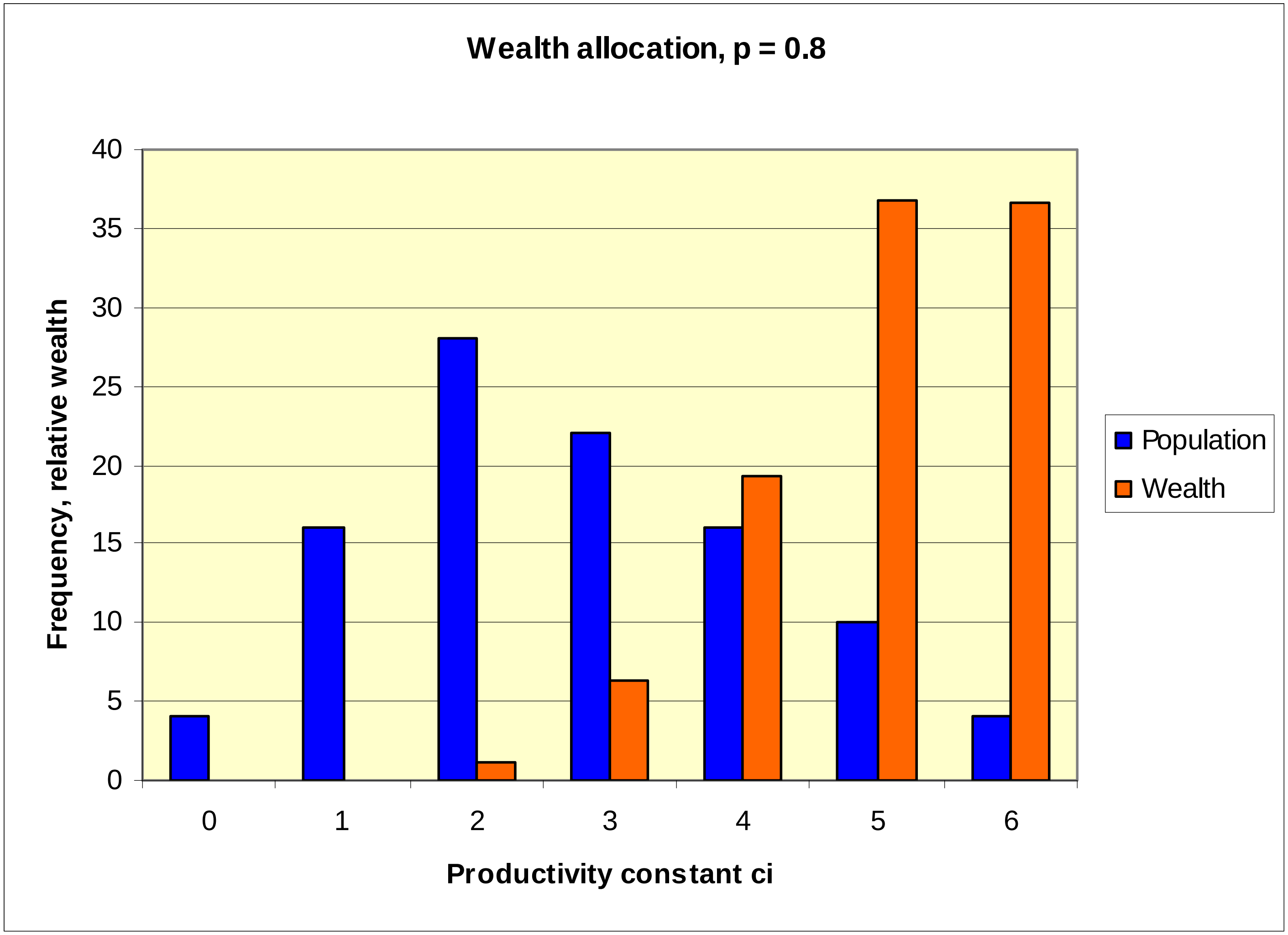

Size of social classes
Frequency
40
35
30
25
20
15
10
5
0
0
1
2
3
4
5
6
Productivity constant ci
Population

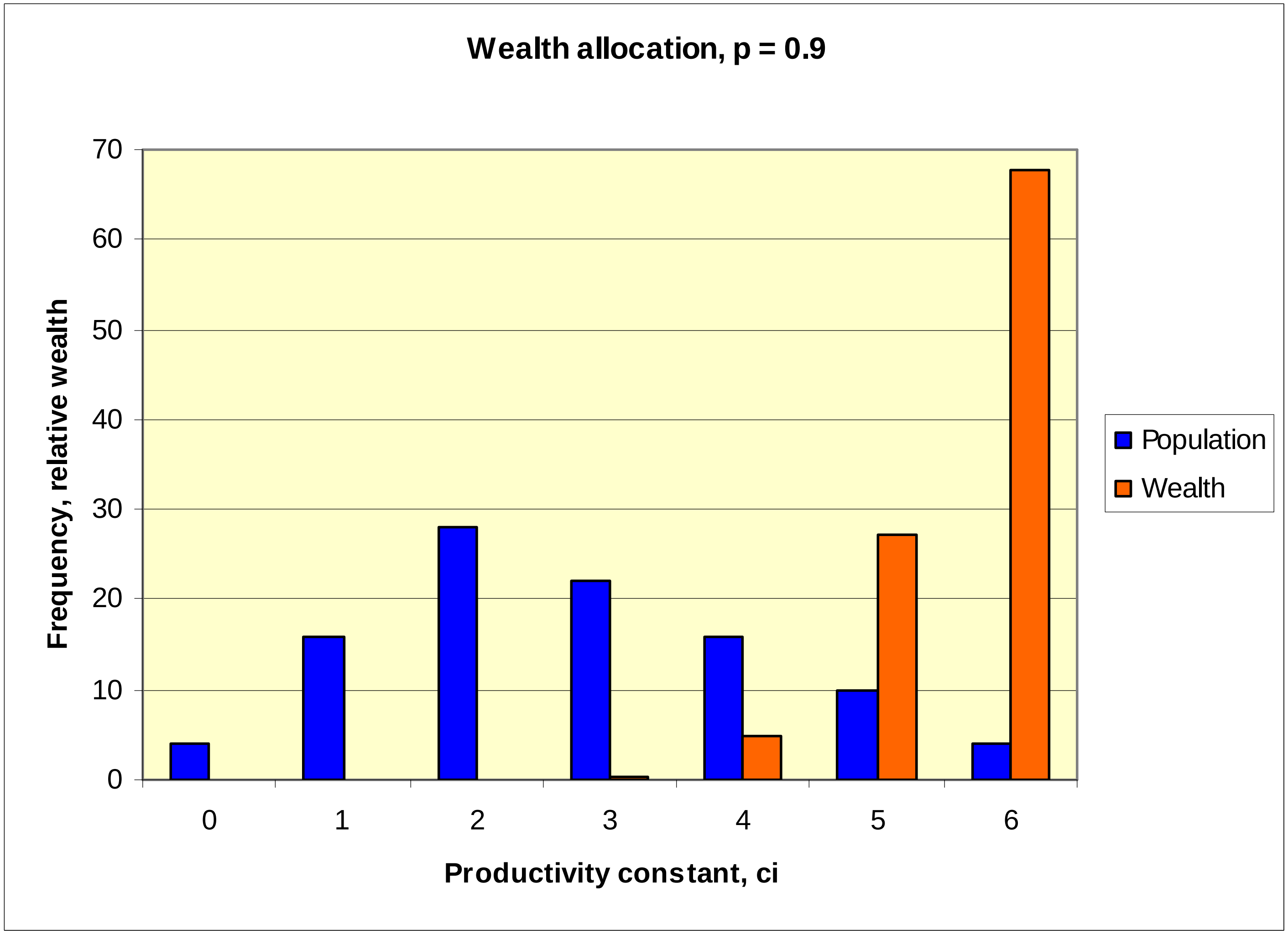

Wealth allocation, p = 0
Frequency, relative wealth
40
35
30
25
20
15
10
5
0
0
1
2
3
4
5
6
Productivity constant ci
Population
Wealth

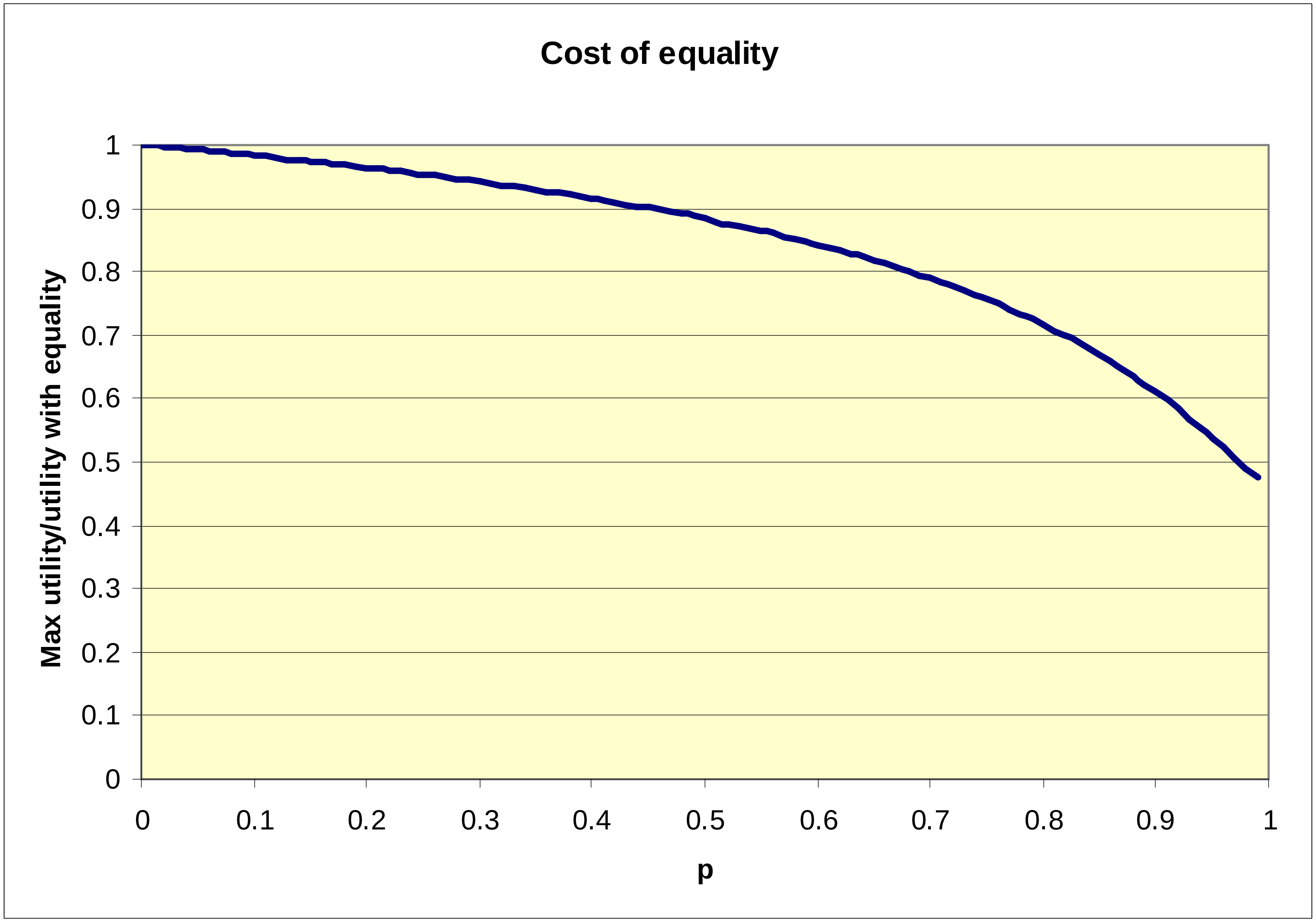

Wealth allocation, p = 0.5
40
35
30
25
20
15
10
5
0
Frequency, relative wealth
0
1
2
3
4
5
6
Productivity constant ci
Population
Wealth



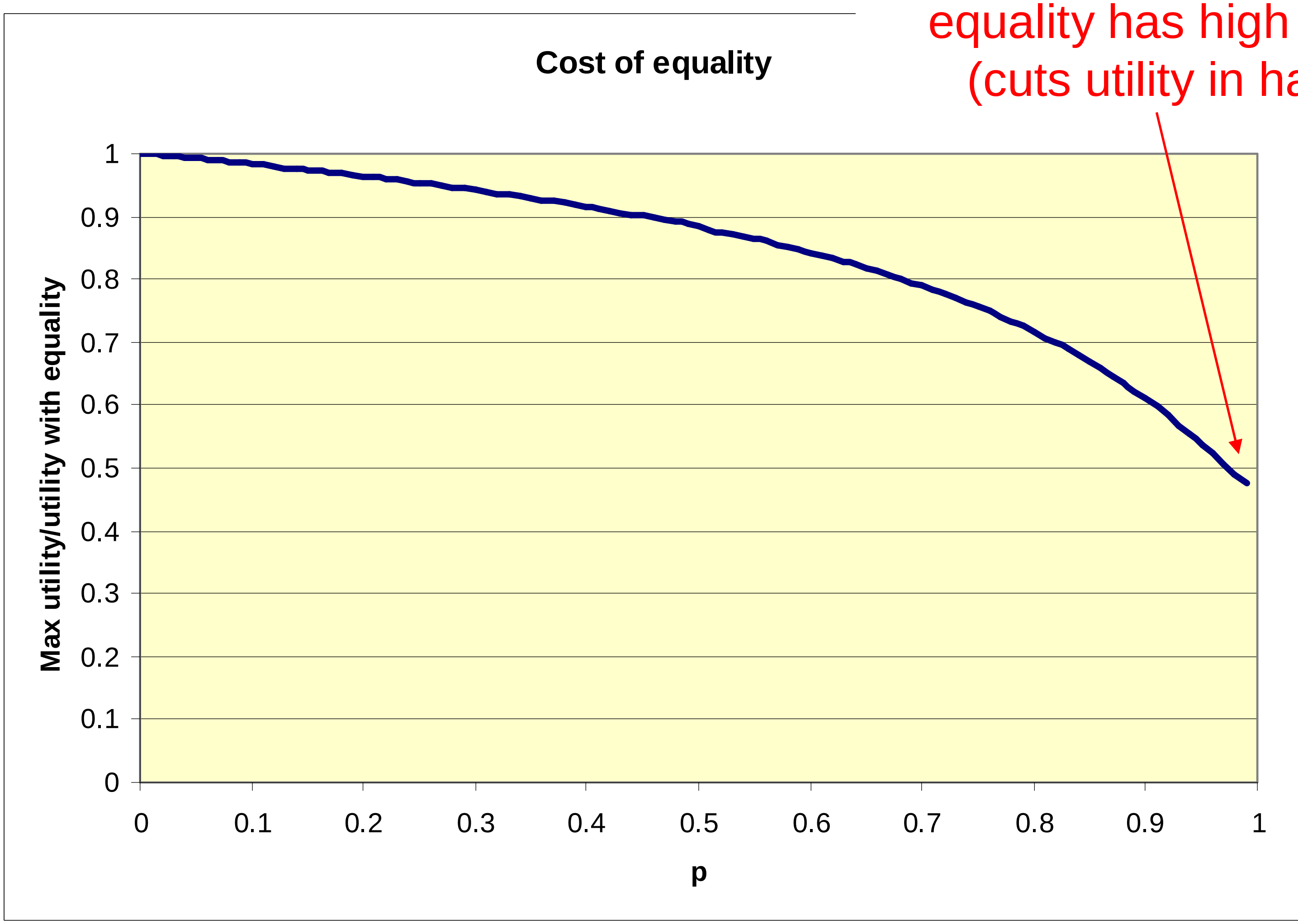
Wealth allocation, p = 0.7
Frequency, relative wealth
40
35
30
25
20
15
10
5
0
0
1
2
3
4
5
6
Productivity constant ci
Population
Wealth


Wealth allocation, p = 0.8
40
35
30
25
20
15
10
5
0
Frequency, relative wealth
0
1
2
3
4
5
6
Productivity constant ci
Population
Wealth

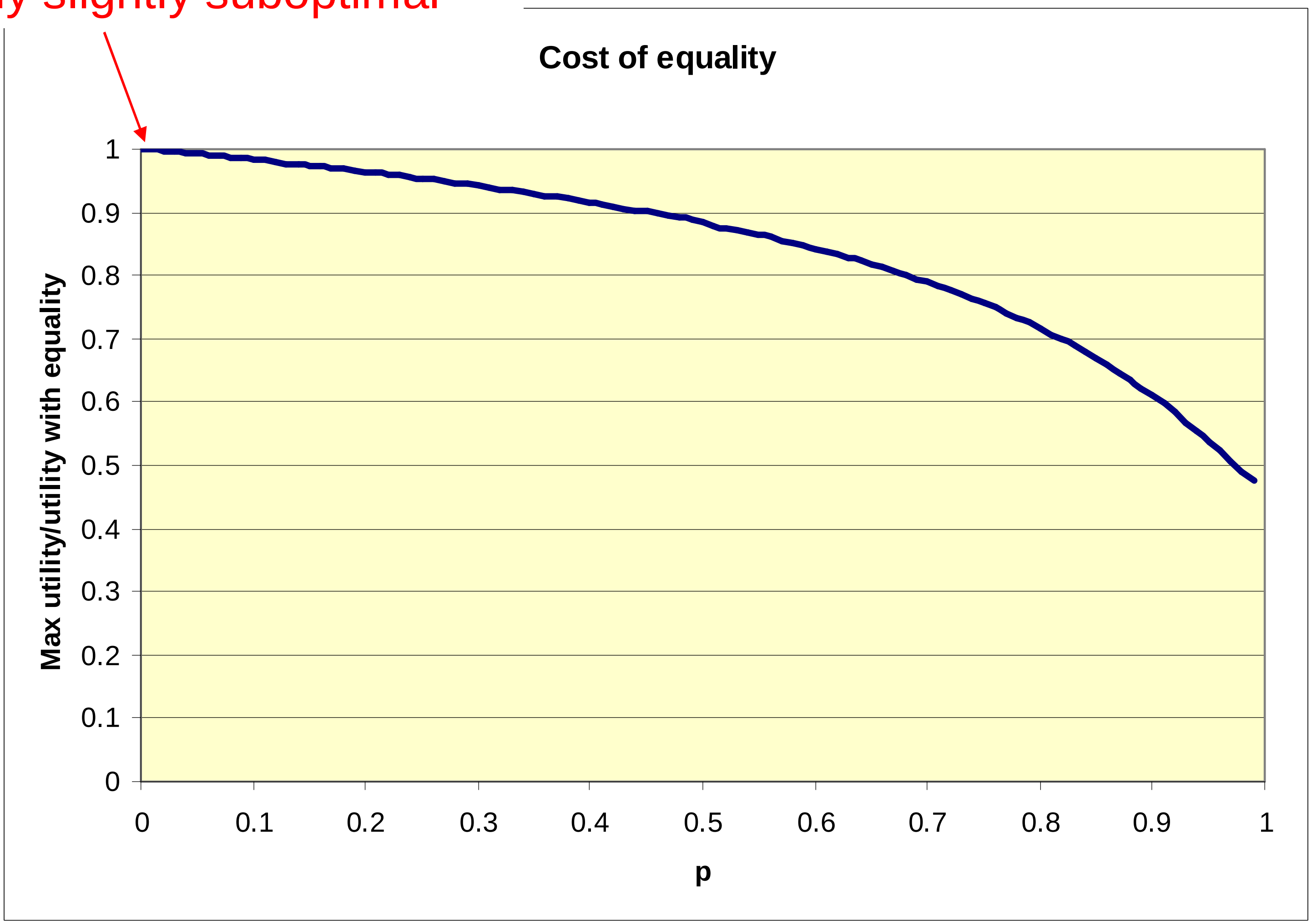

Wealth allocation, p = 0.9
Frequency, relative wealth
70
60
50
40
30
20
10
0
0
1
2
3
4
5
6
Productivity constant, ci
Population
Wealth


Cost of equality
Max utility/utility with equality
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
p

When output is proportional to investment, equality has high cost (cuts utility in half)



As $p \rightarrow 0$, optimal utility requires highly unequal allocation, but equal allocation is only slightly suboptimal



Social Disharmony Model

- Utilitarians argue that a highly unequal distribution cannot be optimal, due to social disharmony.
  - Utility is not an additively separable function.

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- Utilitarians argue that a highly unequal distribution cannot be optimal, due to social disharmony.
  - Utility is not an additively separable function.
  - Let's model cost of inequality as proportional to total range of incomes.

- Now maximize utility:

$$\max \sum_{i=1}^{n} u_i(x_i) - \beta\left(\max_i\{x_i\} - \min_i\{x_i\}\right)$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0, \quad \text{all } i$$

Coefficient of social disharmony

Social Disharmony Model

- Does a positive β result in a more egalitarian distribution of wealth?

- How large must β be to force equality in a utility maximizing distribution?

Social Disharmony Model

- **Theorem.** If $u_i'(\cdot) \le u_{i+1}'(\cdot)$, all i

  we can rewrite the model

$$\max \sum_{i=1}^{n} u_i(x_i) - \beta\left(\max_i\{x_i\} - \min_i\{x_i\}\right)$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x_i \ge 0, \quad \text{all } i$$

as

$$\max \sum_{i=1}^{n} u_i(x_i) - \beta\left(\boxed{x_n - x_1}\right)$$
$$\sum_{i=1}^{n} x_i = 1$$
$$\boxed{x_i \le x_{i+1}, \quad i = 1,\ldots,n-1}$$
$$x_i \ge 0, \quad \text{all } i$$

Social Disharmony Model

- **Theorem.** If $u_i'(\cdot) \le u_{i+1}'(\cdot)$, all i

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$$\max \sum_{i=1}^{n} u_i(x_i) - \beta\left(\max_i\{x_i\} - \min_i\{x_i\}\right)$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x_i \ge 0, \quad \text{all } i$$

as

$$\max \sum_{i=1}^{n} u_i(x_i) - \beta\left(x_n - x_1\right)$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x_i \le x_{i+1}, \quad i = 1, \ldots, n-1$$
$$x_i \ge 0, \quad \text{all } i$$

Associate Lagrange multipliers

λ

μ_i

Social Disharmony Model

- The Karush-Kuhn-Tucker (KKT) optimality conditions imply that x is optimal only if there are λ and $\mu_1, \ldots, \mu_{n-1} \geq 0$ such that

$$u_1'(x_1) + \beta - \lambda - \mu_1 = 0$$
$$u_i'(x_i) - \lambda + \mu_{i-1} - \mu_i = 0, \quad i = 2, \ldots, n-1$$
$$u_n'(x_n) - \beta - \lambda + \mu_{n-1} = 0$$

where $\mu_i = 0$ if $x_i < x_{i+1}$ in the solution.

Social Disharmony Model

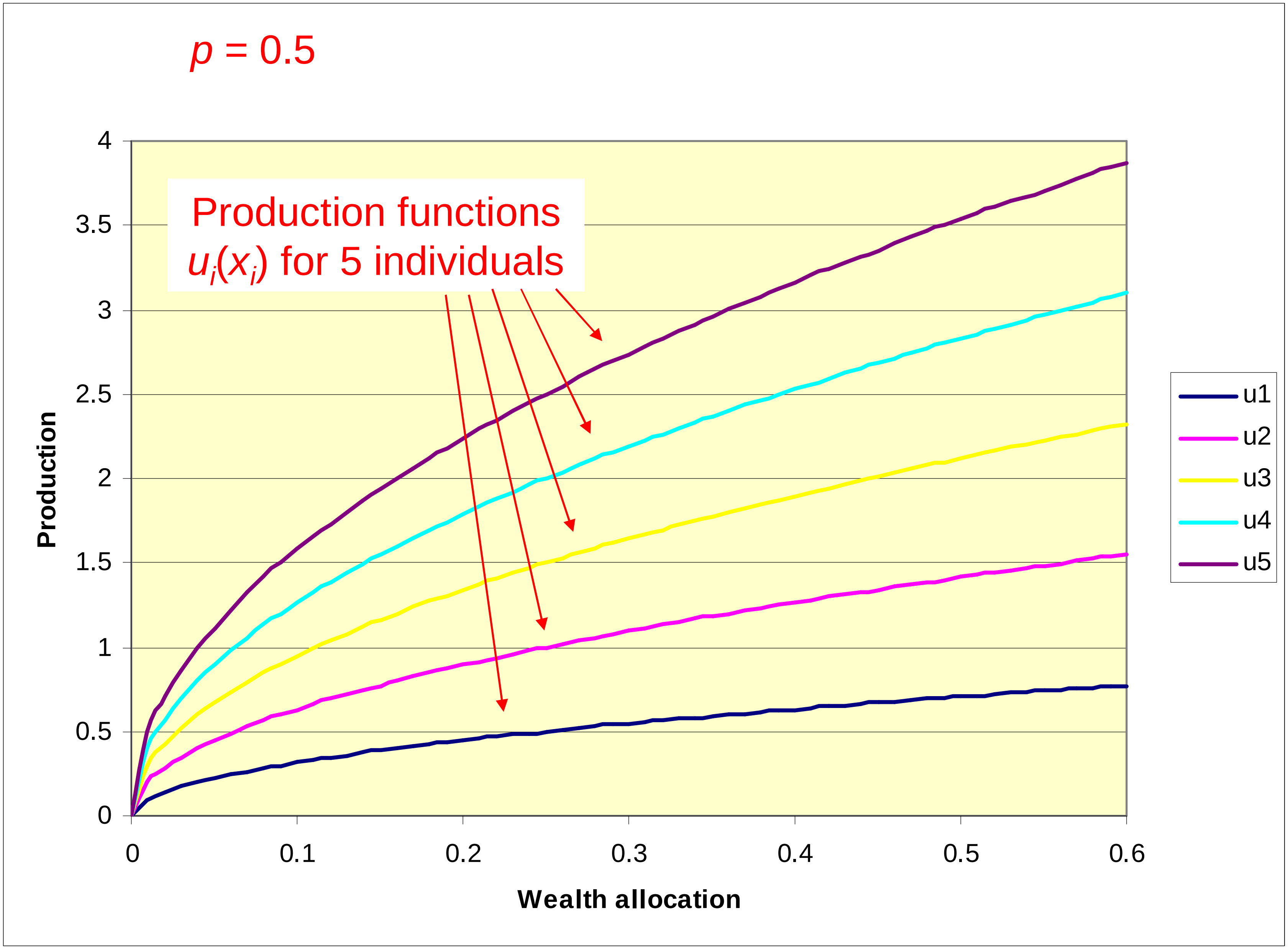
- First suppose that everyone gets a different wealth allotment x_i. Then each $\mu_i = 0$ and

$$\underbrace{u_1'(x_1)}_{\beta \text{ smaller than those in the middle}} \quad \underbrace{u_2'(x_2) \quad \cdots \quad u_{n-1}'(x_{n-1})}_{\text{All equal}} \quad \underbrace{u_n'(x_n)}_{\beta \text{ larger than those in the middle}}$$

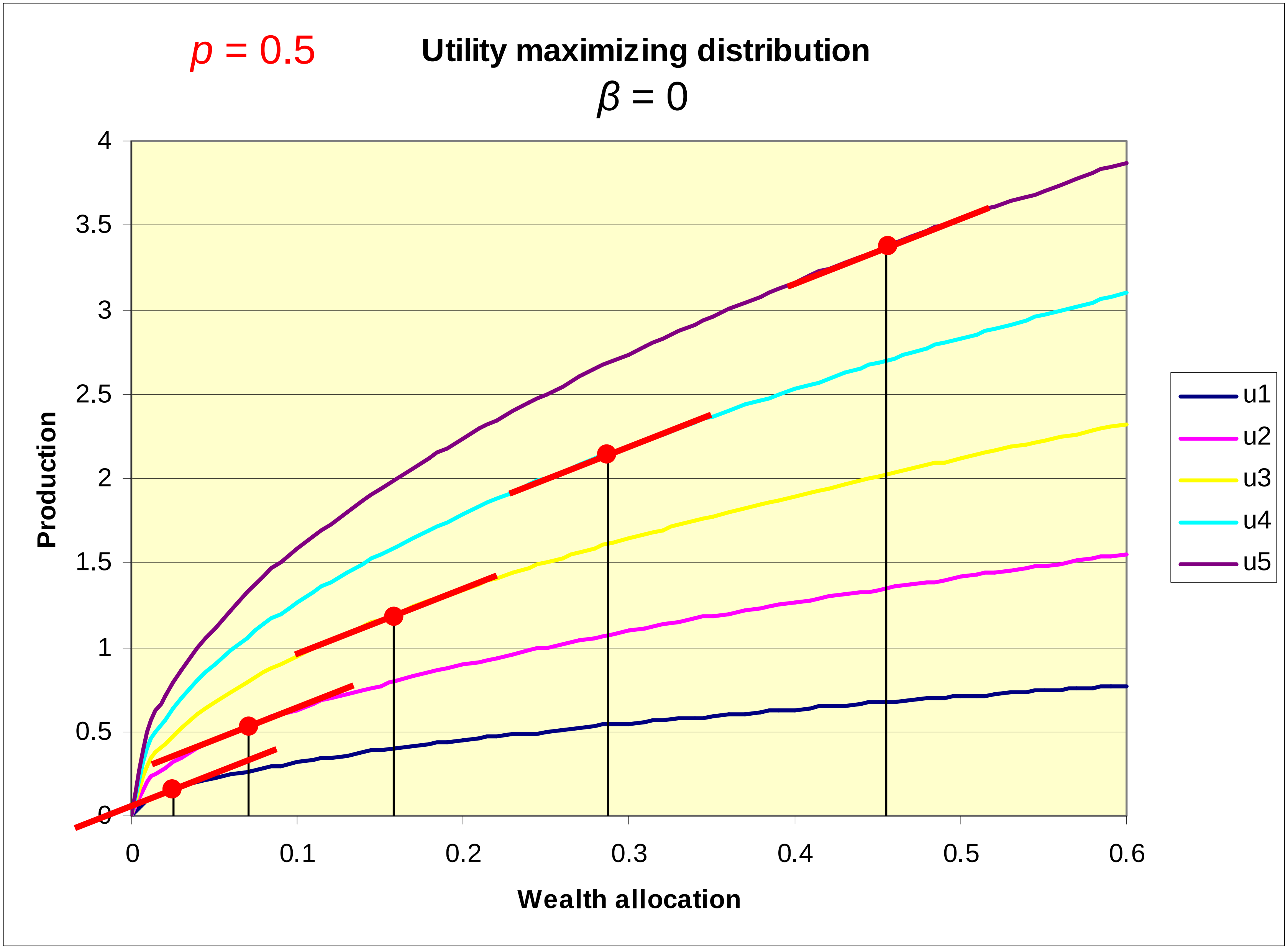
β smaller than those in the middle

All equal

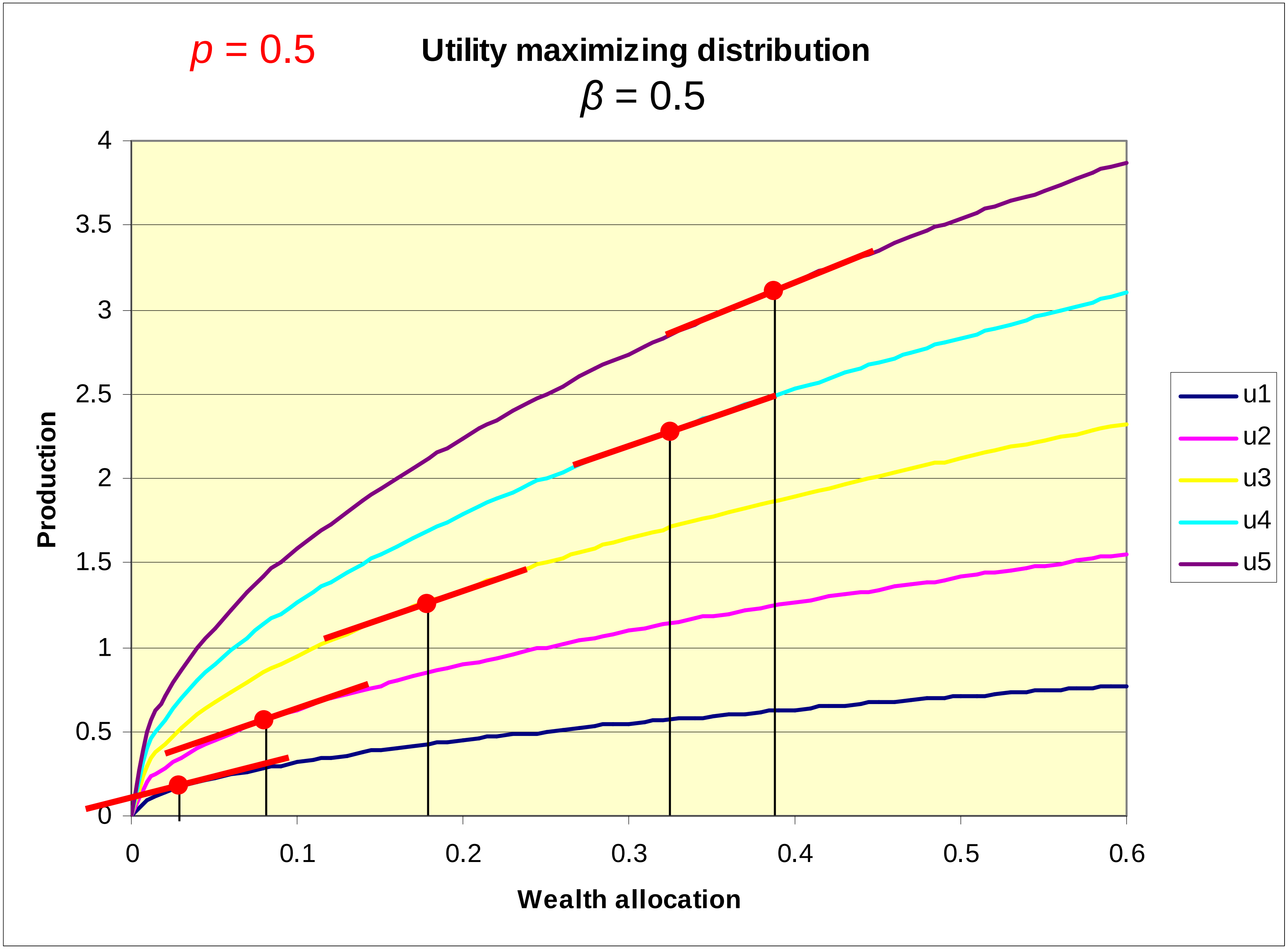
β larger than those in the middle



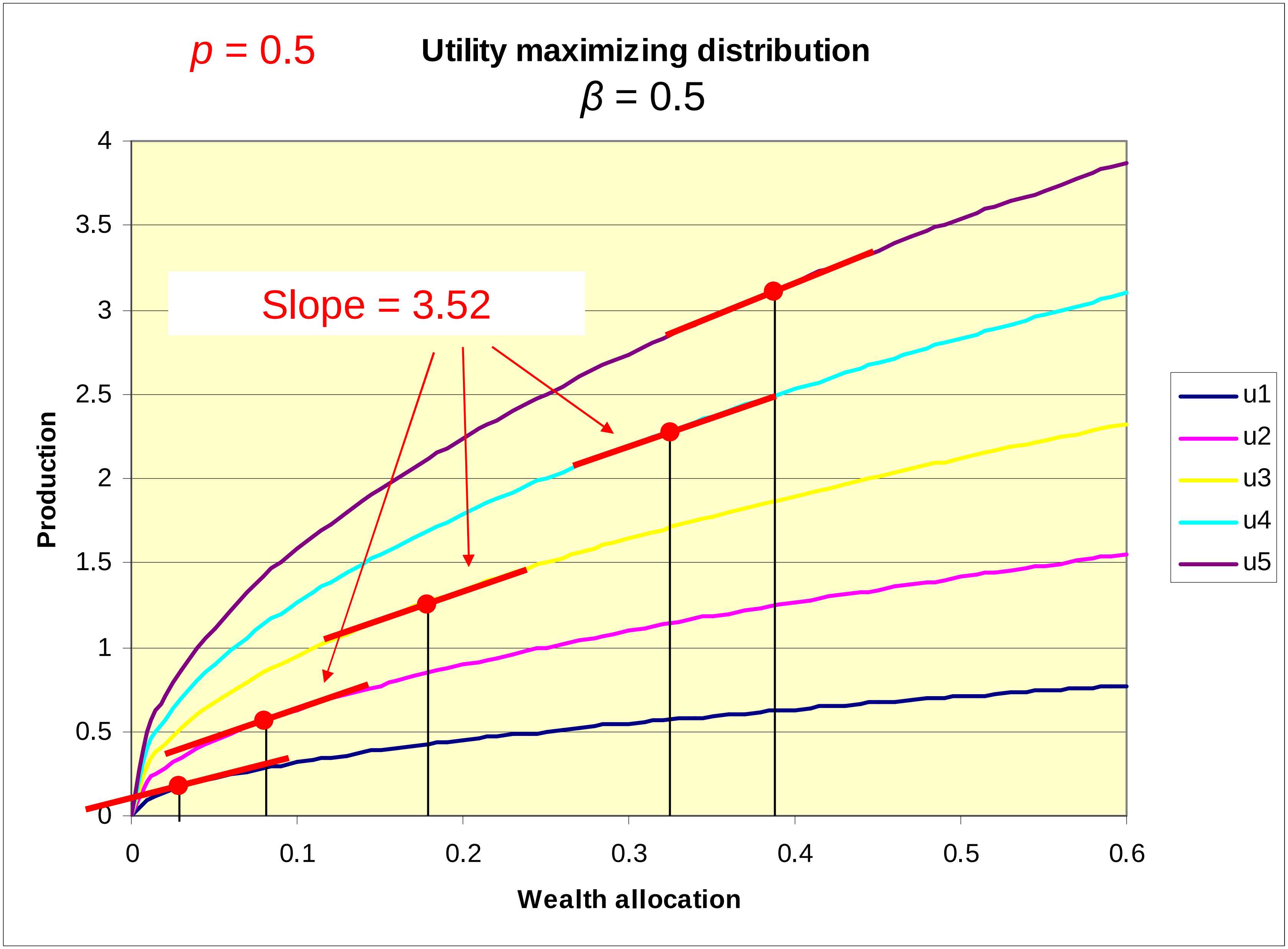
p = 0.5
Production functions
ui(xi) for 5 individuals
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
Production
Wealth allocation
u1
u2
u3
u4
u5



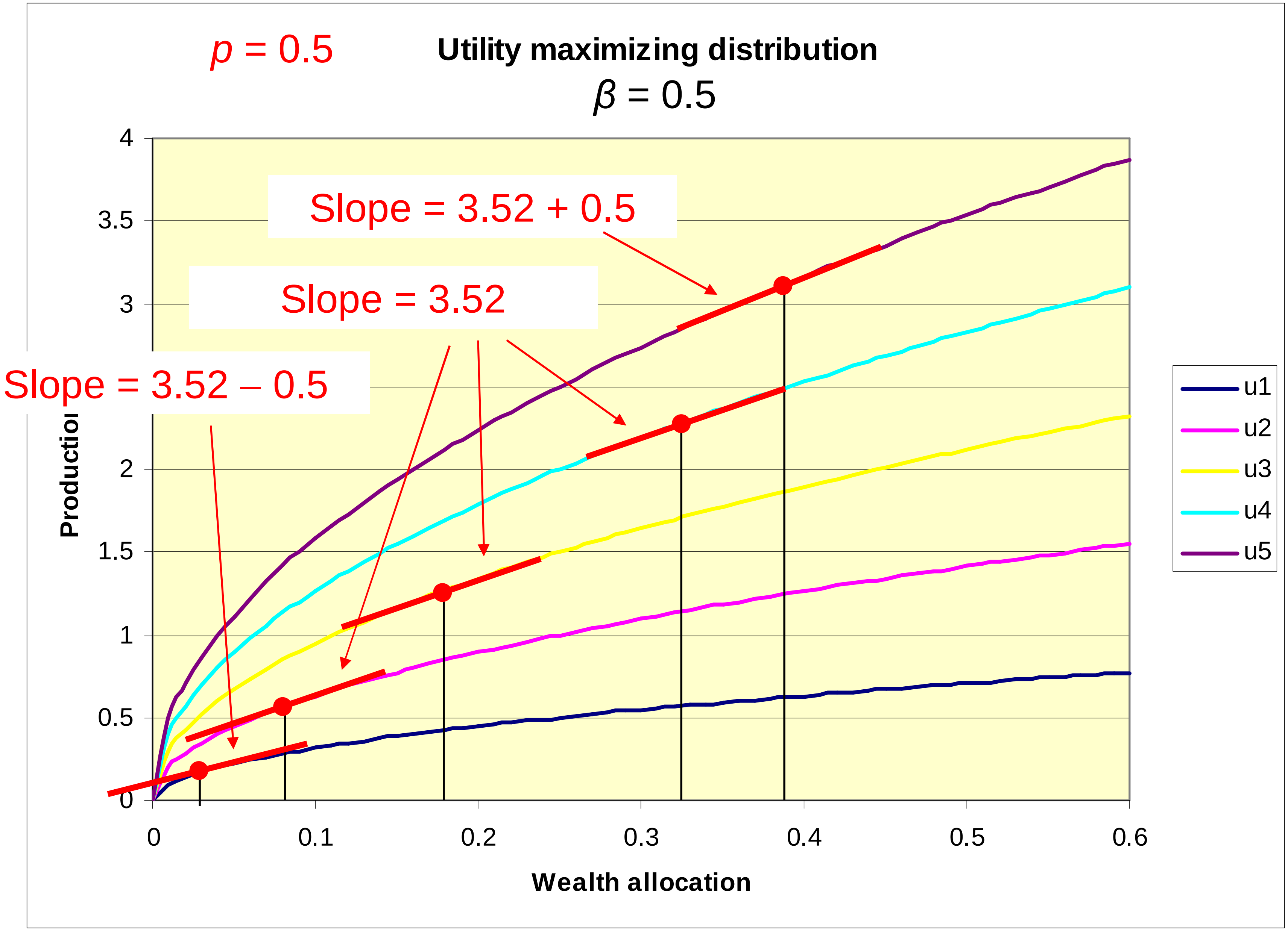
p = 0.5
Utility maximizing distribution
β = 0
Production
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
Wealth allocation
u1
u2
u3
u4
u5



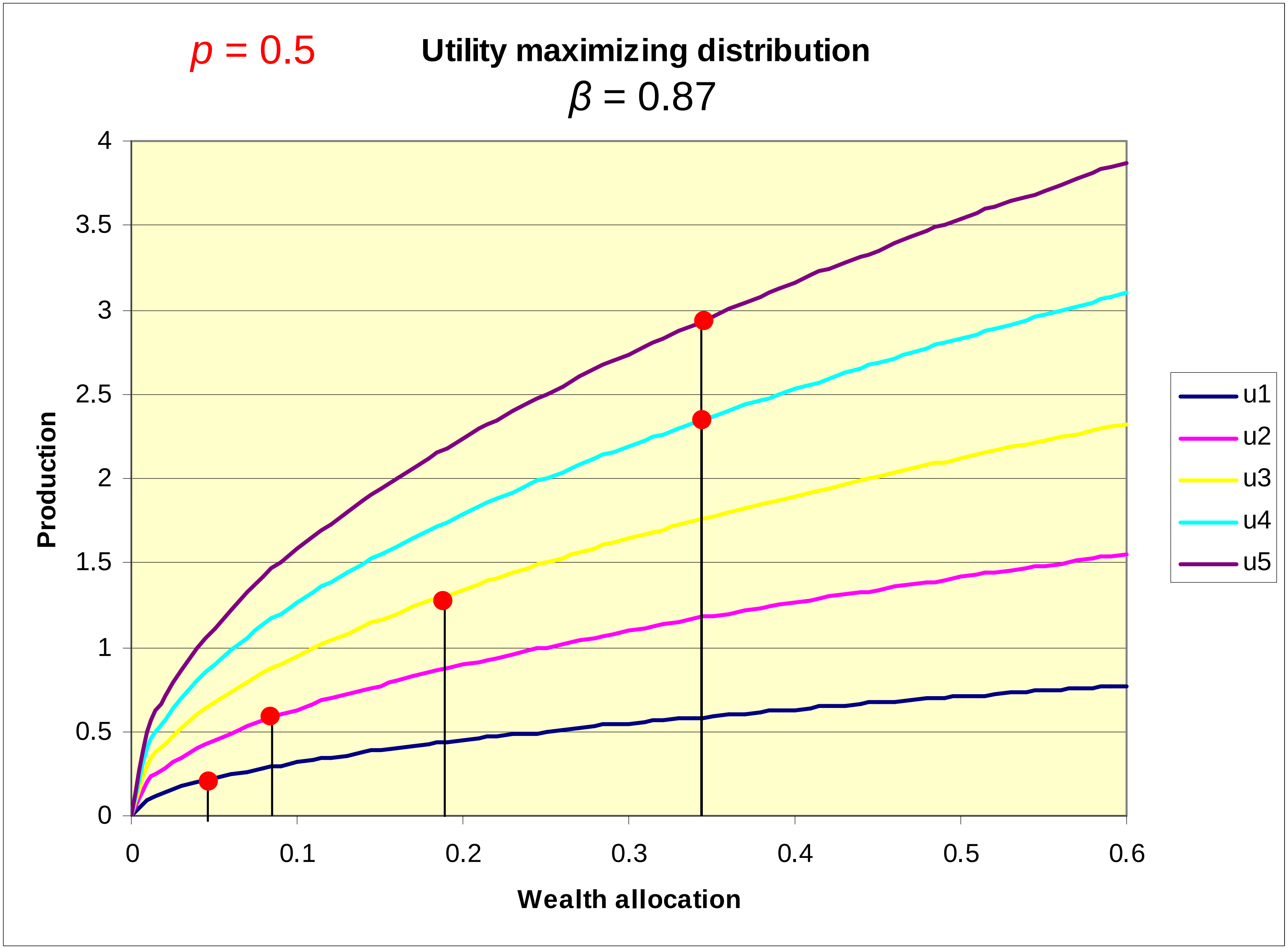
p = 0.5
Utility maximizing distribution
β = 0.5
Production
Wealth allocation
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u1
u2
u3
u4
u5



p = 0.5
Utility maximizing distribution
β = 0.5
Slope = 3.52
Production
Wealth allocation
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u1
u2
u3
u4
u5



p = 0.5
Utility maximizing distribution
β = 0.5
Slope = 3.52 + 0.5
Slope = 3.52
Slope = 3.52 – 0.5
Production
Wealth allocation
0
0.5
1
1.5
2
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u1
u2
u3
u4
u5



p = 0.5
Utility maximizing distribution
β = 0.87
Production
Wealth allocation
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u1
u2
u3
u4
u5

Social Disharmony Model

- How large must β be to force equality?

- Here each $\mu_i > 0$. Eliminate λ from KKT conditions & get equations of the form

$$
\begin{aligned}
&2\mu_1 - \mu_2 = d_1 \\
&\mu_1 + \mu_i - \mu_{i+1} = d_i, \quad i = 2,\ldots,n-2 \\
&\mu_1 + \mu_{n-1} = d_{n-1}
\end{aligned}
$$

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$$
\begin{aligned}
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&\mu_1 + \mu_i - \mu_{i+1} = d_i, \quad i = 2,\ldots,n-2 \\
&\mu_1 + \mu_{n-1} = d_{n-1}
\end{aligned}
$$

- These equations have a particularly simple solution:

$$\mu_k = \frac{k}{n}\sum_{i=k}^{n-1} d_i - \left(1 - \frac{k}{n}\right)\sum_{i=1}^{k-1} d_i$$

Social Disharmony Model

- In this case $$d_i = u_1'(x_1) - u_{i+1}'(x_{i+1}) + \beta, \quad i = 1,\ldots,n-1$$ $$d_{n-1} = u_1'(x_1) - u_n'(x_n) + 2\beta$$

- So,

$$\mu_k = \beta + \frac{k(n-k)}{n}\left(\boxed{\frac{1}{n-k}\sum_{i=k+1}^{n} u_i'(1/n)} - \boxed{\frac{1}{k}\sum_{i=1}^{k} u_i'(1/n)}\right)$$

Average over *n* – *k* most productive individuals

Average over *k* least productive individuals

Social Disharmony Model

- **Theorem.** The utilitarian distribution is egalitarian only if each $\mu_k \geq 0$, thus only if for all k,

$$\beta \geq \frac{k(n-k)}{n}\left(\boxed{\frac{1}{n-k}\sum_{i=k+1}^{n} u_i'(1/n)} - \boxed{\frac{1}{k}\sum_{i=1}^{k} u_i'(1/n)}\right)$$

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Average over k least productive individuals

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If $u_i'(x_i) = c_i x_i^p$, we have equality only if for all k,

$$\beta \geq \frac{p}{n^{p-1}} \cdot \frac{k(n-k)}{n}\left(\boxed{\frac{1}{n-k}\sum_{i=k+1}^{n} c_i} - \boxed{\frac{1}{k}\sum_{i=1}^{k} c_i}\right)$$

Average over $n-k$ most productive individuals

Average over k least productive individuals

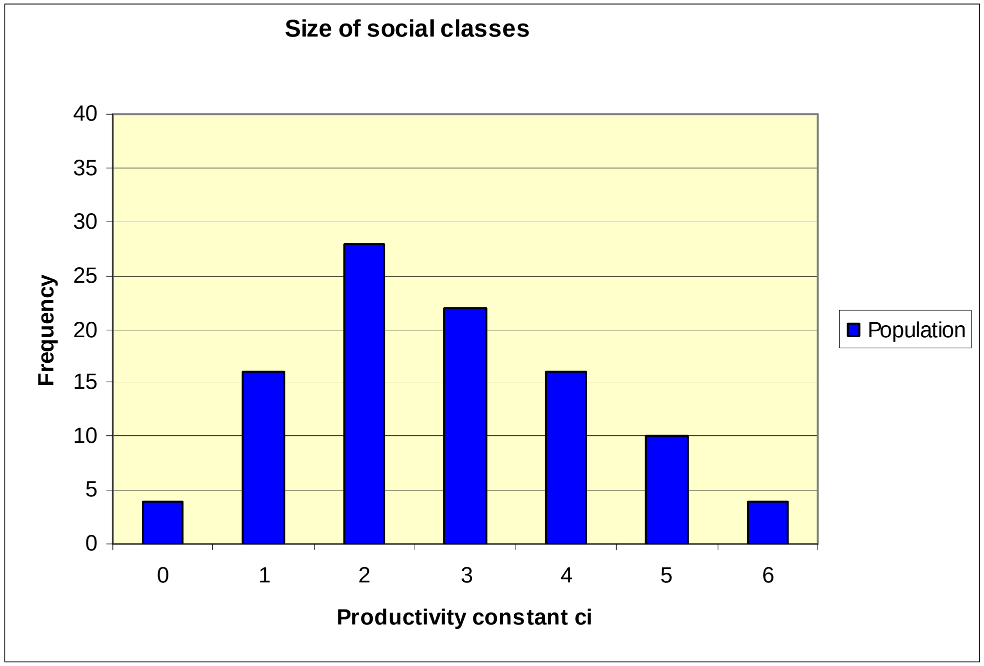
Social Disharmony Model

If $u_i'(x_i) = c_i x_i^p$, we have equality only if for all k,

$$\beta \geq \frac{p}{n^{p-1}} \cdot \frac{k(n-k)}{n} \left(\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \right)$$

- So β must be larger to enforce equality when there is a large gap between k least productive people and the rest.
  - β is more sensitive to the gap when $k \approx n/2$, because $k(n-k)$ is larger.

Unimodal productivity distribution



$c_{avg} = 0$ $\quad$ $c_{avg} = 2.88$

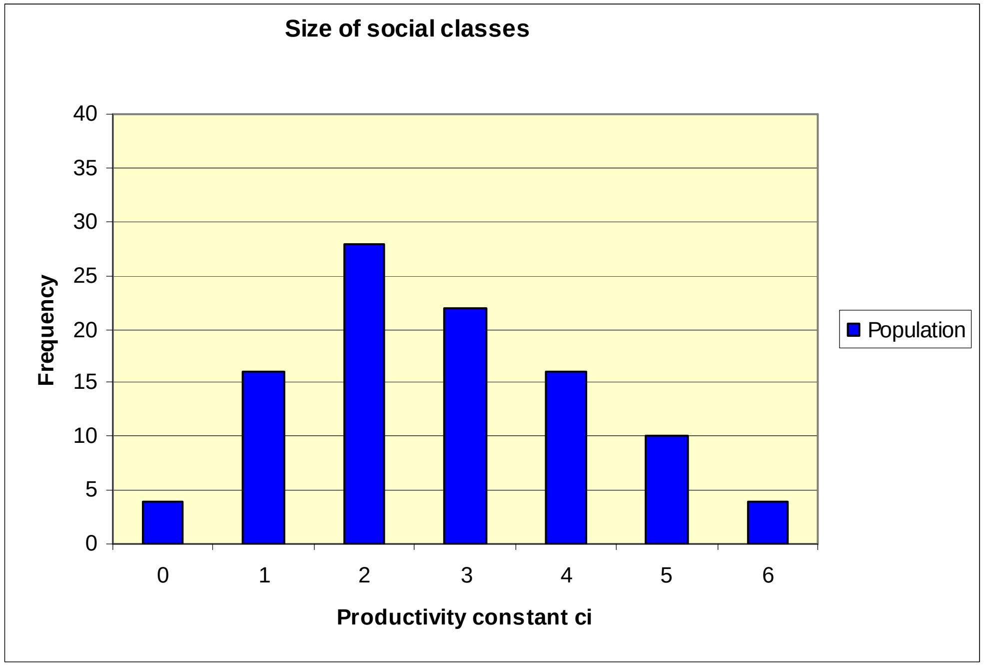
$$\frac{\beta}{U_{max}} \geq 1.77$$

Total utility of egalitarian distribution, ignoring cost of social disharmony

$$\frac{U_{egal}}{U_{max}} = \frac{27.6}{31.2} = 0.88$$

Maximum total utility, ignoring cost of social disharmony

Unimodal productivity distribution



$c_{avg} = 0.8$ $\quad$ $c_{avg} = 3.25$

$$\frac{\beta}{U_{max}} \geq 6.27$$

Total utility of egalitarian distribution, ignoring cost of social disharmony

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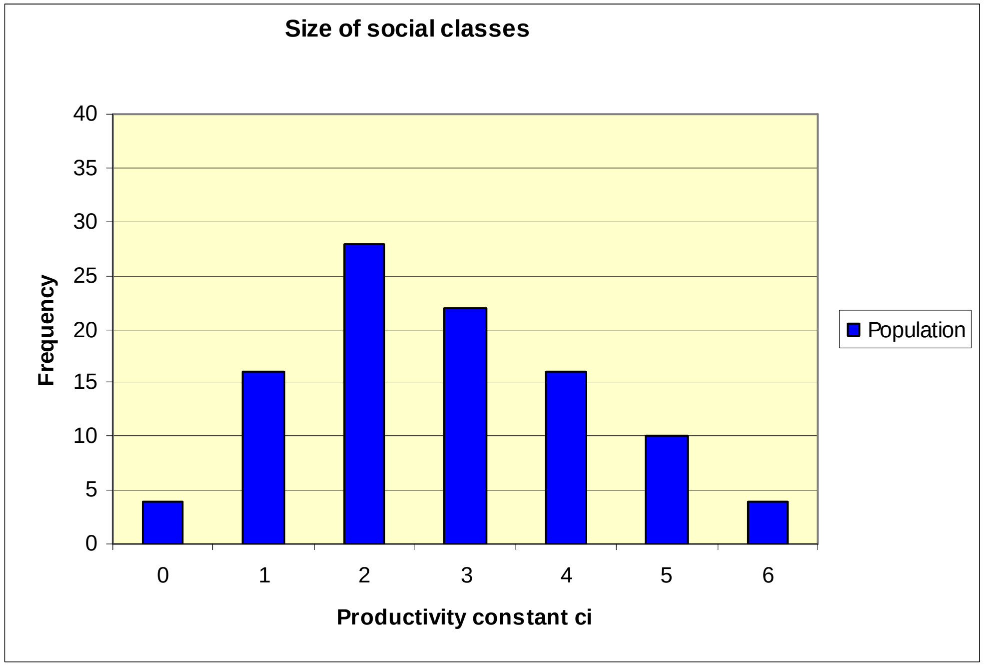
Maximum total utility, ignoring cost of social disharmony

Unimodal productivity distribution



$c_{avg} = 1.5$ $\quad$ $c_{avg} = 3.92$

$$\frac{\beta}{U_{max}} \geq 9.68$$

Total utility of egalitarian distribution, ignoring cost of social disharmony

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Maximum total utility, ignoring cost of social disharmony

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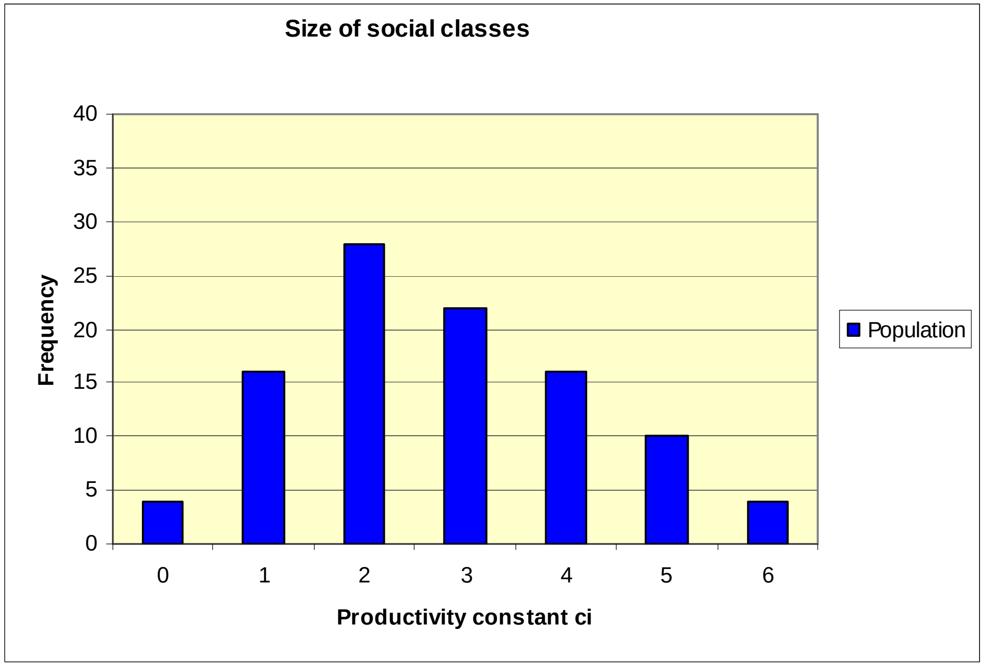


$c_{avg} = 1.97$ $c_{avg} = 4.6$

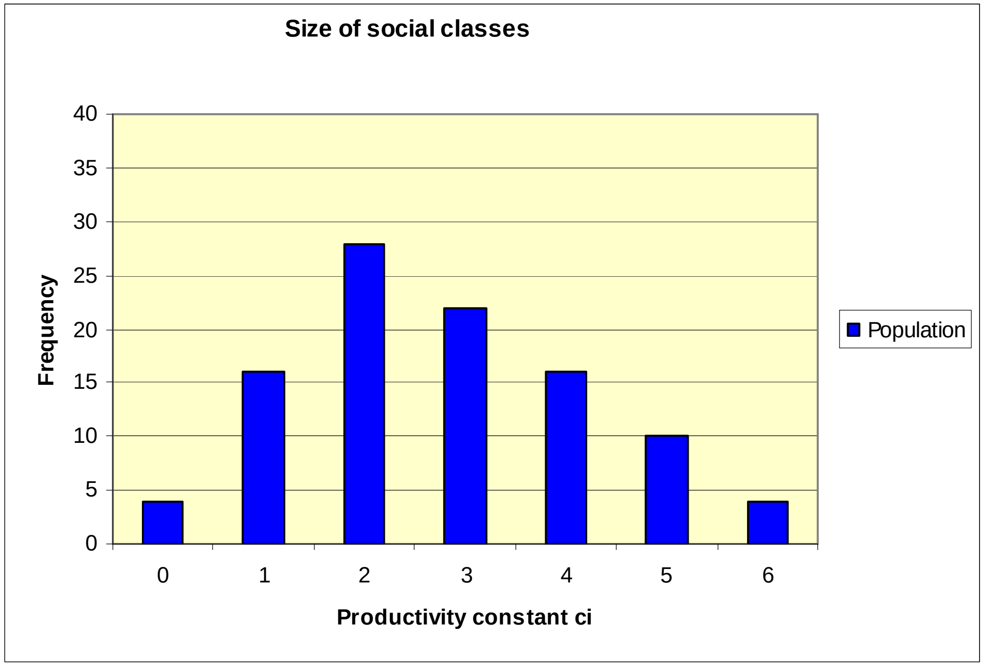
$$\frac{\beta}{U_{\max}} \geq 8.83$$

Total utility of egalitarian distribution, ignoring cost of social disharmony

$$\frac{U_{\text{egal}}}{U_{\max}} = \frac{27.6}{31.2} = 0.88$$

Maximum total utility, ignoring cost of social disharmony

Unimodal productivity distribution



$c_{avg} = 2.35$ $\quad$ $c_{avg} = 5.29$

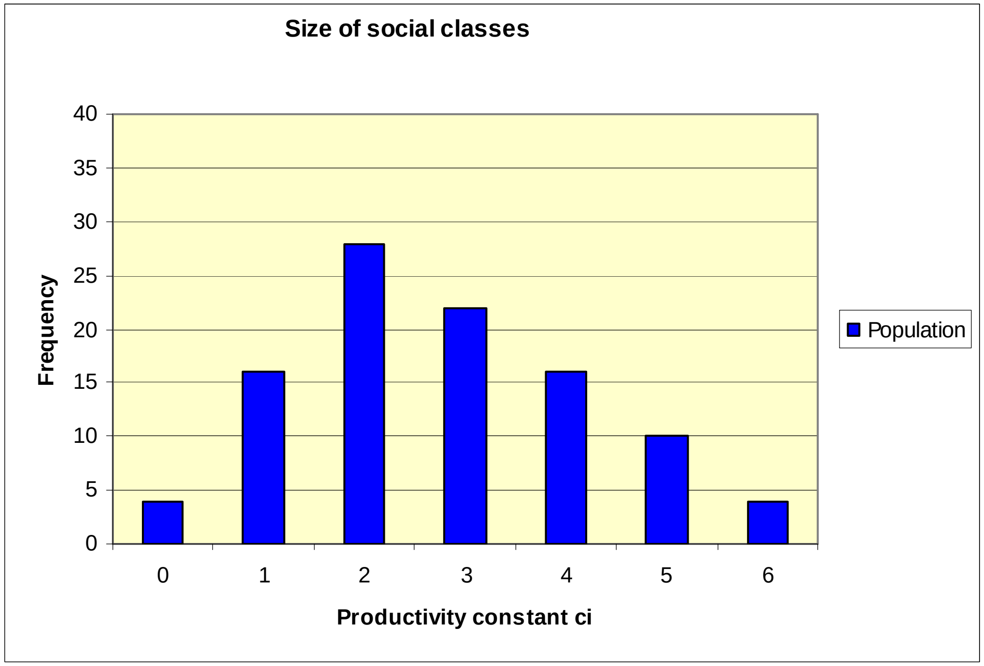
$$\frac{\beta}{U_{max}} \geq 5.66$$

Total utility of egalitarian distribution, ignoring cost of social disharmony

$$\frac{U_{egal}}{U_{max}} = \frac{27.6}{31.2} = 0.88$$

Maximum total utility, ignoring cost of social disharmony

Unimodal productivity distribution



c_{avg}= 2.63 c_{avg}= 6

$$\frac{\beta}{U_{max}} \geq 2.07$$

Total utility of egalitarian distribution, ignoring cost of social disharmony

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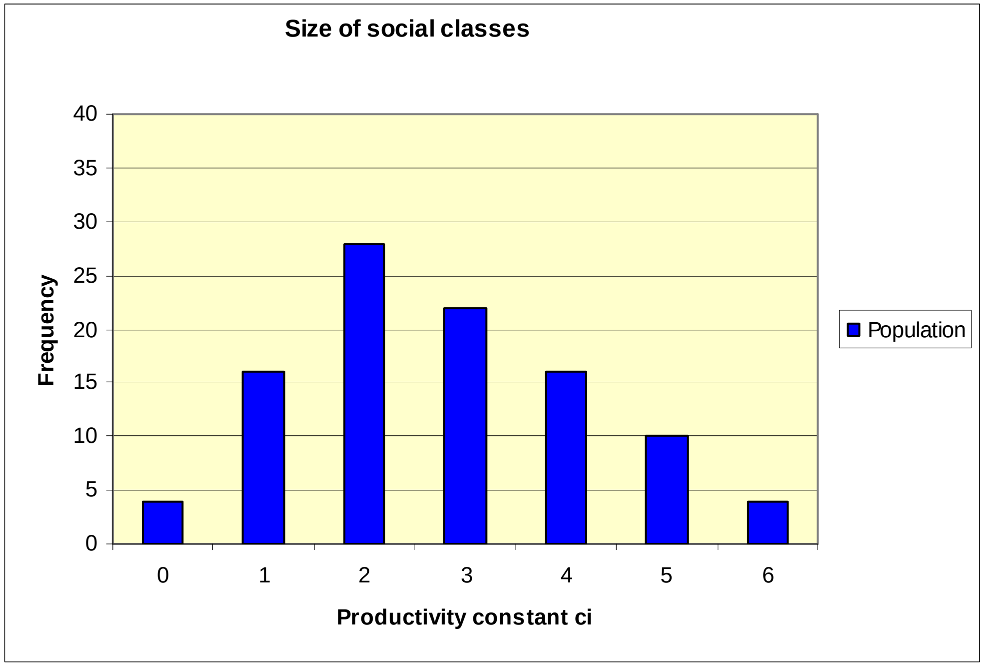
Maximum total utility, ignoring cost of social disharmony

Unimodal productivity distribution



Total utility of egalitarian distribution, ignoring cost of social disharmony

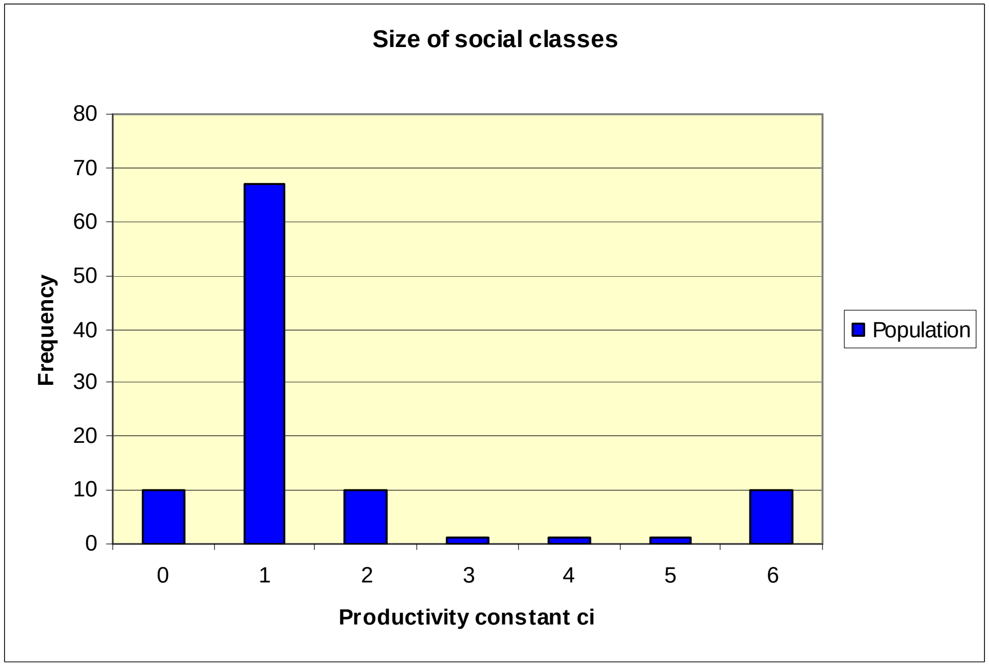
$$\frac{U_{\text{egal}}}{U_{\max}} = \frac{27.6}{31.2} = 0.88$$

Maximum total utility, ignoring cost of social disharmony

To enforce equality, let

$$\frac{\beta}{U_{\max}} \geq 9.68$$

Bimodal productivity distribution

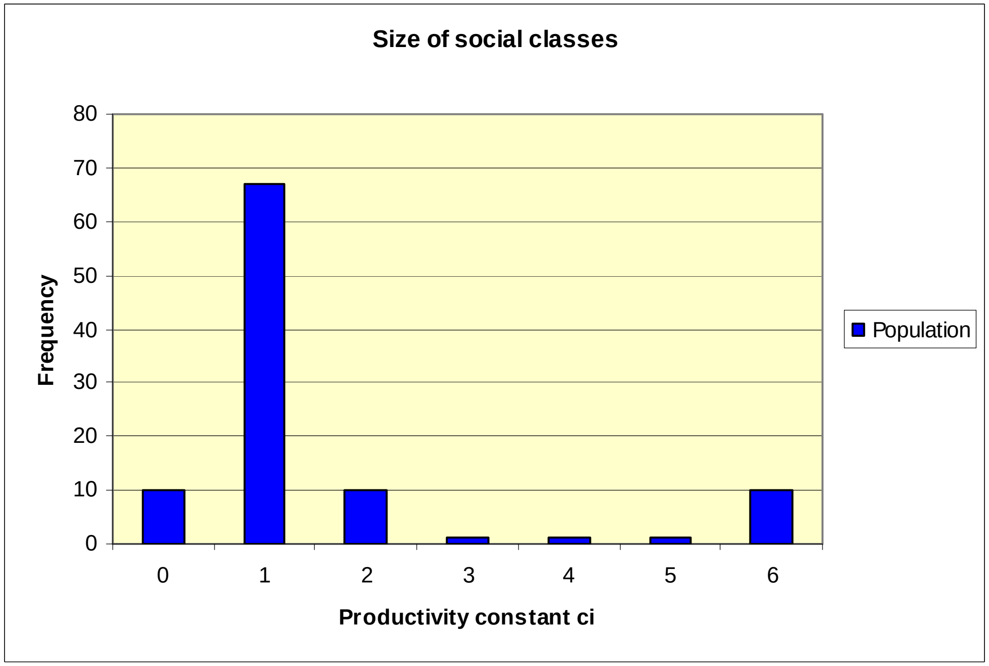


$c_{avg} = 0$ $c_{avg} = 1.77$

$$\frac{U_{egal}}{U_{max}} = \frac{15.9}{22.7} = 0.70$$

$$\frac{\beta}{U_{max}} \geq 3.50$$

Bimodal productivity distribution

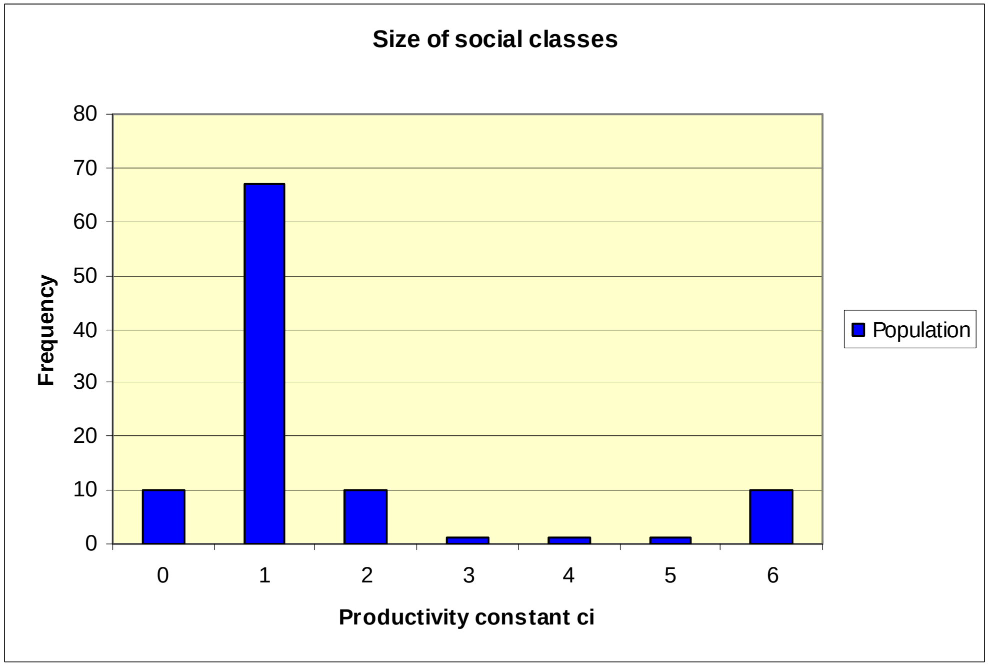


$c_{avg} = 0.87$ $c_{avg} = 4$

$$\frac{U_{egal}}{U_{max}} = \frac{15.9}{22.7} = 0.70$$

$$\frac{\beta}{U_{max}} \geq 12.19$$

Bimodal productivity distribution

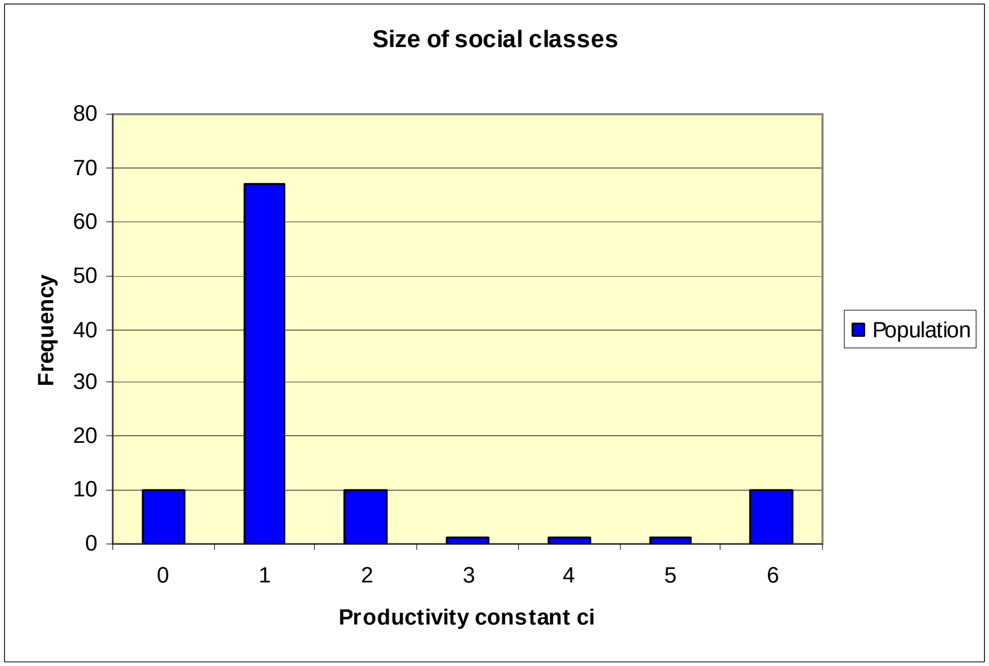


$c_{avg} = 1$ $\quad$ $c_{avg} = 5.54$

$$\frac{U_{egal}}{U_{max}} = \frac{15.9}{22.7} = 0.70$$

$$\frac{\beta}{U_{max}} \geq 11.29$$

Bimodal productivity distribution



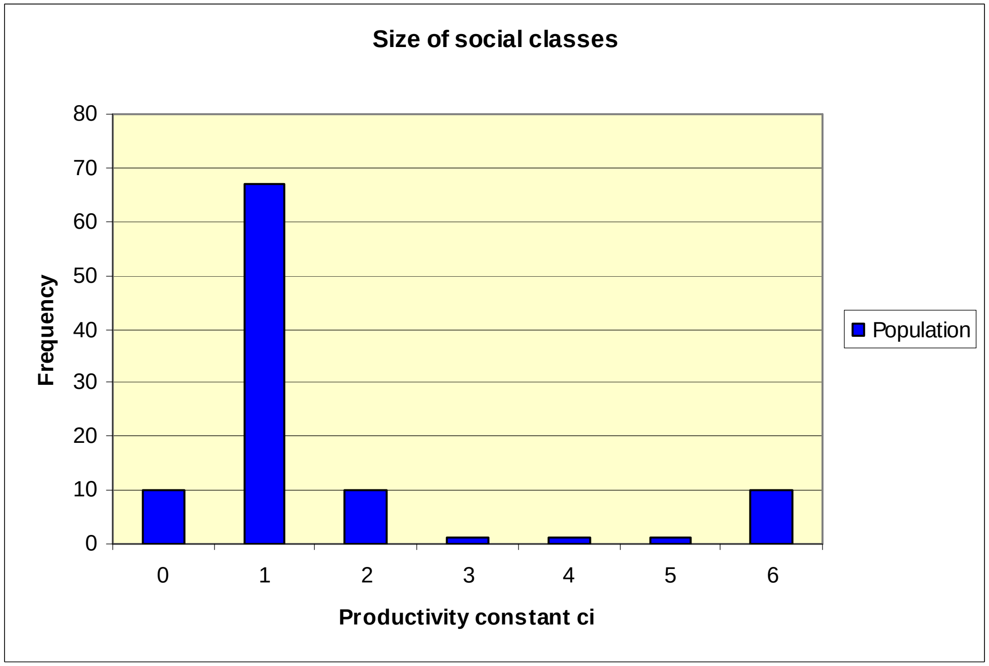
$$\frac{U_{egal}}{U_{max}} = \frac{15.9}{22.7} = 0.70$$

c_{avg}= 1.02

c_{avg}= 5.75

$$\frac{\beta}{U_{max}} \geq 10.98$$

Bimodal productivity distribution

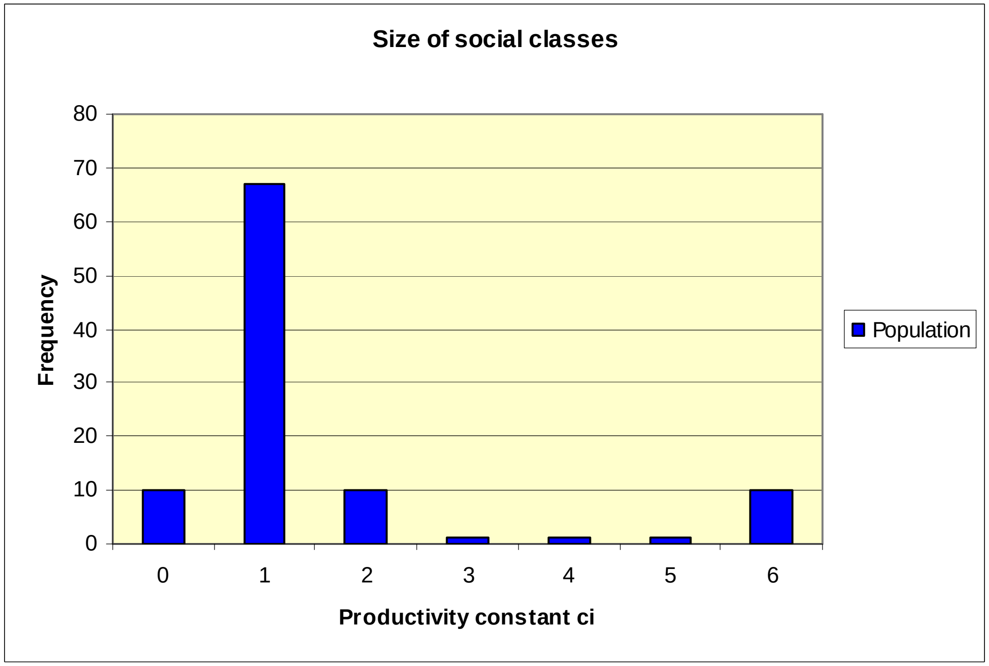


$c_{avg} = 1.06$ $c_{avg} = 5.91$

$$\frac{U_{egal}}{U_{max}} = \frac{15.9}{22.7} = 0.70$$

$$\frac{\beta}{U_{max}} \geq 10.45$$

Bimodal productivity distribution

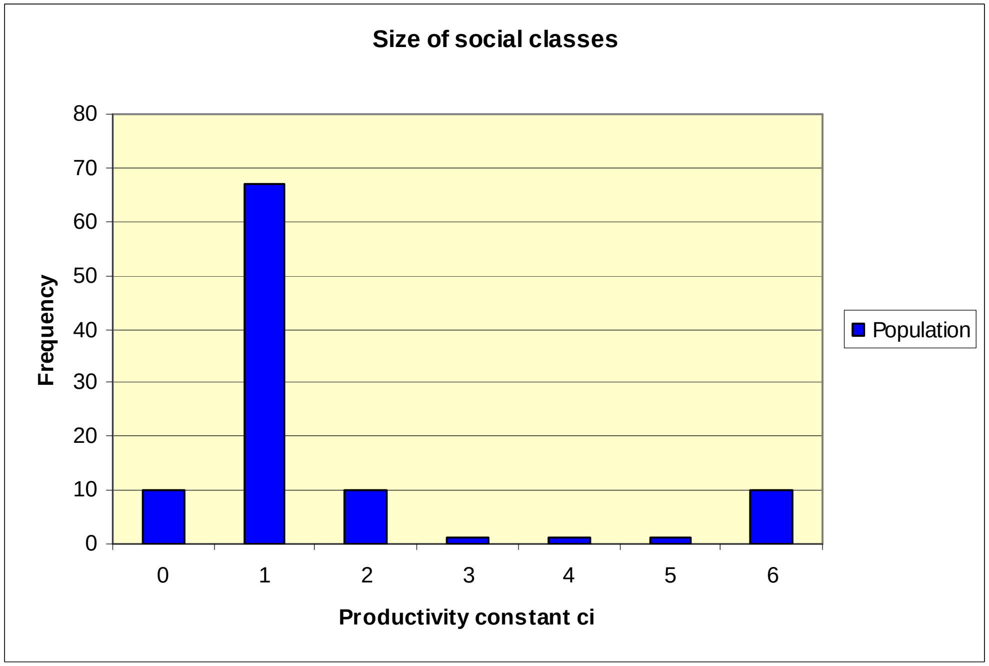


$c_{avg} = 1.1$ $c_{avg} = 6$

$$\frac{U_{egal}}{U_{max}} = \frac{15.9}{22.7} = 0.70$$

$$\frac{\beta}{U_{max}} \geq 9.70$$

Bimodal productivity distribution



$$\frac{U_{egal}}{U_{max}} = \frac{15.9}{22.7} = 0.70$$

To enforce equality, let

$$\frac{\beta}{U_{max}} \geq 12.19$$

Rawlsian Model

- **Rawlsian difference principle:** A just distribution of wealth creates only as much inequality as is necessary to improve **everyone's** welfare.
  - This refers to inequality of **opportunity**, not outcome.
  - As in distribution of salaries, tax burden, medical benefits, etc.

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  - This refers to inequality of **opportunity**, not outcome.
  - As in distribution of salaries, tax burden, medical benefits, etc.

- This is a **lexmax** principle.
  - Maximize welfare of least advantaged class…
  - then next to least advantaged class…
  - and so forth.

Rawlsian Model

- We will model this as a **utility maximization problem** with a **lexmax objective function**.

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- Let $v(x_i)$ = personal utility of wealth x_i.
  - We assume everyone has the same utility function, but not the same productivity function.

Rawlsian Model

- We will model this as a **utility maximization problem** with a **lexmax objective function**.

- Let $v(x_i)$ = personal utility of wealth x_i.
  - We assume everyone has the same utility function, but not the same productivity function.

- We assume each person's share of total utility is **proportional** to the utility of his/her initial wealth allocation.
  - Thus the initial allocation affects social privileges for life.

Rawlsian Model

- The utility maximization problem:

$$\begin{aligned}
&\operatorname{lexmax}\ (y_1,\ldots,y_n)\\
&\frac{y_i}{y_1}=\frac{v(x_i)}{v(x_1)},\quad i=2,\ldots,n\\
&\sum_{i=1}^{n} y_i=\sum_{i=1}^{n} u_i(x_i)\\
&\sum_{i=1}^{n} x_i=1\\
&\boxed{x_i}\geq 0,\ \ \text{all } i
\end{aligned}$$

Wealth allocation to person *i*

Rawlsian Model

- The utility maximization problem:

$$\text{lexmax}\ \ (y_1, \ldots, y_n)$$

Utility allocation to person *i*

$$\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n$$

$$\sum_{i=1}^{n} \boxed{y_i} = \sum_{i=1}^{n} u_i(x_i)$$

Wealth allocation to person *i*

$$\sum_{i=1}^{n} x_i = 1$$

$$\boxed{x_i} \geq 0, \quad \text{all } i$$

Rawlsian Model

- The utility maximization problem:

Utility allocation to person *i*

Wealth allocation to person *i*

Budget

$$\text{lexmax} \ (y_1, \ldots, y_n)$$

$$\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n$$

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yi's sum to total utility produced

Wealth allocation to person *i*

$$\sum_{i=1}^{n} x_i = 1$$

Budget

$$\boxed{x_i} \geq 0, \quad \text{all } i$$

Rawlsian Model

- The utility maximization problem:

Proportional allocation of total utility

Utility allocation to person *i*

$$\text{lexmax}\ \ (y_1,\ldots,y_n)$$

$$\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2,\ldots,n$$

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$$\operatorname{lexmax}\ (y_1,\ldots,y_n)$$

$$\boxed{\begin{aligned} &\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2,\ldots,n \\ &\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i) \\ &\sum_{i=1}^{n} x_i = 1 \\ &x_i \geq 0, \ \text{all } i \end{aligned}}$$

To define lexmax:

Let L_k be the problem of maximizing

$$\min\{y_k,\ldots,y_n\}$$

subject to this and

$$(y_1,\ldots,y_{k-1}) = (y_1^*,\ldots,y_{k-1}^*)$$

Optimal solution of L_{k-1}

Rawlsian Model

- The utility maximization problem:

$$\operatorname{lexmax}\ (y_1,\ldots,y_n)$$

$$\boxed{\begin{aligned}
&\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2,\ldots,n \\
&\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i) \\
&\sum_{i=1}^{n} x_i = 1 \\
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\end{aligned}}$$

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Then y^* solves lexmax problem if $(y_1{}^*,\ldots,y_k{}^*)$ solves L_k for $k = 1,\ldots,n$.

Rawlsian Model

- The utility maximization problem:

$$\text{lexmax}\ \ (y_1,\ldots,y_n)$$

$$\boxed{\begin{aligned} &\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2,\ldots,n \\ &\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i) \\ &\sum_{i=1}^{n} x_i = 1 \\ &x_i \geq 0, \ \ \text{all } i \end{aligned}}$$

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$$\min\{y_k,\ldots,y_n\}$$

subject to this and

$$(y_1,\ldots,y_{k-1}) = (y_1{}^*,\ldots,y_{k-1}{}^*)$$

Note: The literature often defines L_k to maximize y_k --not this

Rawlsian Model

- The utility maximization problem:

$$\begin{array}{|l|}\hline \text{lexmax } (y_1, \ldots, y_n) \\ \dfrac{y_i}{y_1} = \dfrac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n \\ \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i) \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0, \quad \text{all } i \\ \hline \end{array}$$

Theorem. If $u_i'(\cdot) \leq u_{i+1}'(\cdot)$ and $v(\cdot)$ is nondecreasing, this has an optimal solution in which $y_1 \leq \cdots \leq y_n$

Rawlsian Model

- So L_k is

$$\max \min\{y_k,\ldots,y_n\}$$

$$(x_1,\ldots,x_{k-1})=(x_1^*,\ldots,x_{k-1}^*)$$

$$\frac{y_i}{y_1}=\frac{v(x_i)}{v(x_1)}$$

$$\sum_{i=1}^{n} y_i=\sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i=1$$

$$x_i \ge 0,\quad i=1,\ldots,k-1$$

Rawlsian Model

- So L_k is

$$\max \boxed{y_k}$$

$$(y_1,\ldots,y_{k-1}) = (y_1*,\ldots,y_{k-1}*)$$

$$\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\boxed{y_k \le \cdots \le y_n}$$

$$x_i \ge 0, \quad i = 1,\ldots,k-1$$

Apply the theorem

Rawlsian Model

- So L_k is

$$\max \boxed{v(x_k)\frac{\sum_{i=1}^{n} u_i(x_i)}{\sum_{i=1}^{n} v(x_i)}}$$

$$\boxed{(x_1,\ldots,x_{k-1}) = (x_1^*,\ldots,x_{k-1}^*)}$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\boxed{x_k \le \cdots \le x_n}$$

$$x_k \ge 0$$

Eliminate y_i's

Rawlsian Model

- When does the Rawlsian model result in equality?
  - That is, when do we have $x_1 = \cdots = x_n$ in the solution of the lexmax problem?

- The lexmax problem forces equality if and only if L_1 forces equality.

Rawlsian Model

- L_1 is $$\max v(x_1)\frac{\sum_{i=1}^{n} u_i(x_i)}{\sum_{i=1}^{n} v(x_i)}$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_1 \leq \cdots \leq x_n$$

$$x_k \geq 0$$

Rawlsian Model

- L_1 is $$\max v(x_1)\frac{\sum_{i=1}^{n} u_i(x_i)}{\sum_{i=1}^{n} v(x_i)}$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_1 \le \cdots \le x_n$$

$$x_k \ge 0$$

Associate Lagrange multipliers $\mu_1, \ldots, \mu_{n-1}$

Rawlsian Model

- Remarkably, the KKT conditions have the **same form** as for the social disharmony model:

$$\begin{aligned}
&2\mu_1 - \mu_2 = d_1 \\
&\mu_1 + \mu_i - \mu_{i+1} = d_i, \quad i = 2,\ldots,n-2 \\
&\mu_1 + \mu_{n-1} = d_{n-1}
\end{aligned}$$

where in this case

$$d_i = v(x_i)\frac{\sum_i c_i u_i(x_i)}{\sum_i v(x_i)}\left(\frac{v'(x_1)}{v(x_1)} - \frac{u_{i+1}{}'(x_{i+1}) - u_1{}'(x_1)}{\sum_i c_i u_i(x_i)} + \frac{v'(x_{i+1}) - v'(x_1)}{\sum_i v(x_i)}\right)$$

Rawlsian Model

- **Theorem.** If $u_i(x_i) = c_i x_i^p$ and $v(x_i) = x_i^q$, then the lexmax distribution is egalitarian ($x_1 = \cdots = x_n$) only if

$$\frac{1}{k}\sum_{i=1}^{k}\left(1+\frac{q}{p}\cdot\frac{n-k}{k}\right)c_i \geq \frac{1}{n-k}\sum_{i=k+1}^{n}\left(1-\frac{q}{p}\cdot\frac{n-k}{k}\right)c_i$$

for $k = 1, \ldots, n-1$.

Rawlsian Model

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for $k = 1, \ldots, n - 1$.

Average of k smallest c_i's, each augmented by

$$\frac{q}{p}\cdot\frac{n-k}{k}$$

Average of $n - k$ largest c_i's, each reduced by

$$\frac{q}{p}\cdot\frac{n-k}{k}$$

Rawlsian Model

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for $k = 1, \ldots, n-1$.

- Equality is **more likely** when p is small.
  - That is, when greater investment in an individual yields rapidly decreasing marginal returns.

Rawlsian Model

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$$\frac{1}{\boxed{k}}\sum_{i=1}^{k}\left(1+\frac{q}{p}\cdot\frac{n-k}{\boxed{k}}\right)c_i \geq \frac{1}{n-\boxed{k}}\sum_{i=k+1}^{n}\left(1-\frac{q}{p}\cdot\frac{n-k}{\boxed{k}}\right)c_i$$

for $k = 1, \ldots, n-1$.

- Equality test is **more sensitive** at upper end (large k).
  - Equality is **unlikely** when individuals at the top are much more productive than average.
  - Equality is **still possible** even when individuals at the bottom are much less productive than the average.

Rawlsian Model

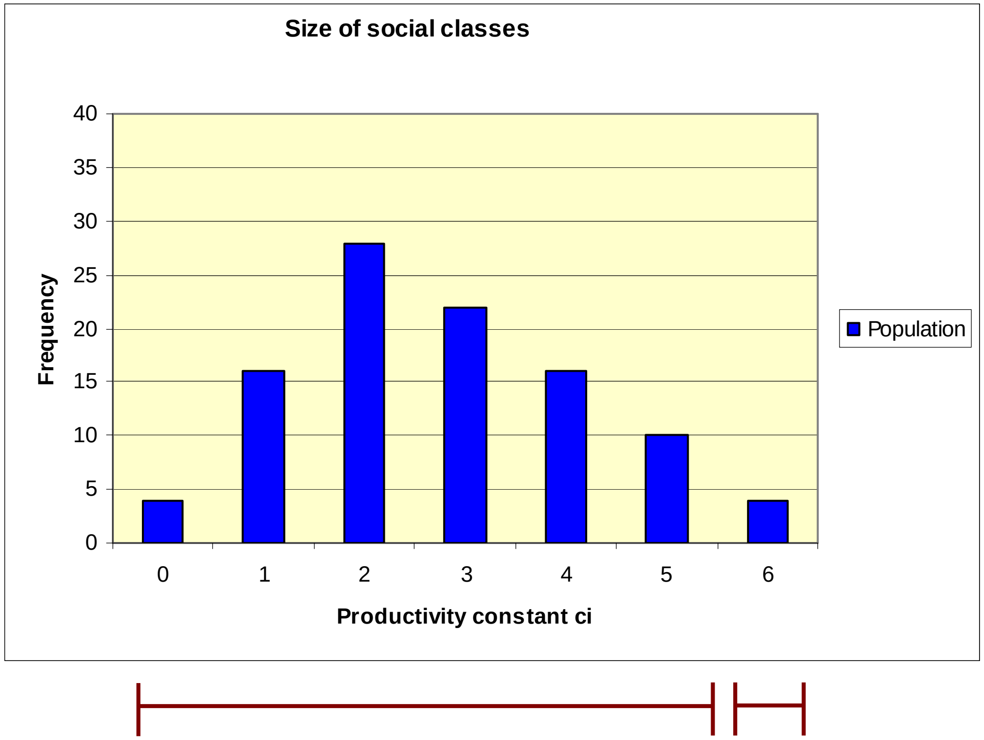
- **Theorem.** If $u_i(x_i) = c_i x_i^p$ and $v(x_i) = x_i^q$, then the lexmax distribution is egalitarian ($x_1 = \cdots = x_n$) only if

$$\frac{1}{k}\sum_{i=1}^{k}\left(1+\frac{\boxed{q}}{p}\cdot\frac{n-k}{k}\right)c_i \geq \frac{1}{n-k}\sum_{i=k+1}^{n}\left(1-\frac{\boxed{q}}{p}\cdot\frac{n-k}{k}\right)c_i$$

for $k = 1, \ldots, n-1$.

- Equality is **less likely** when q is small.
  - That is, when greater wealth yields rapidly decreasing marginal utility.
  - That is, when people **don't care much about getting rich**.

Unimodal productivity distribution

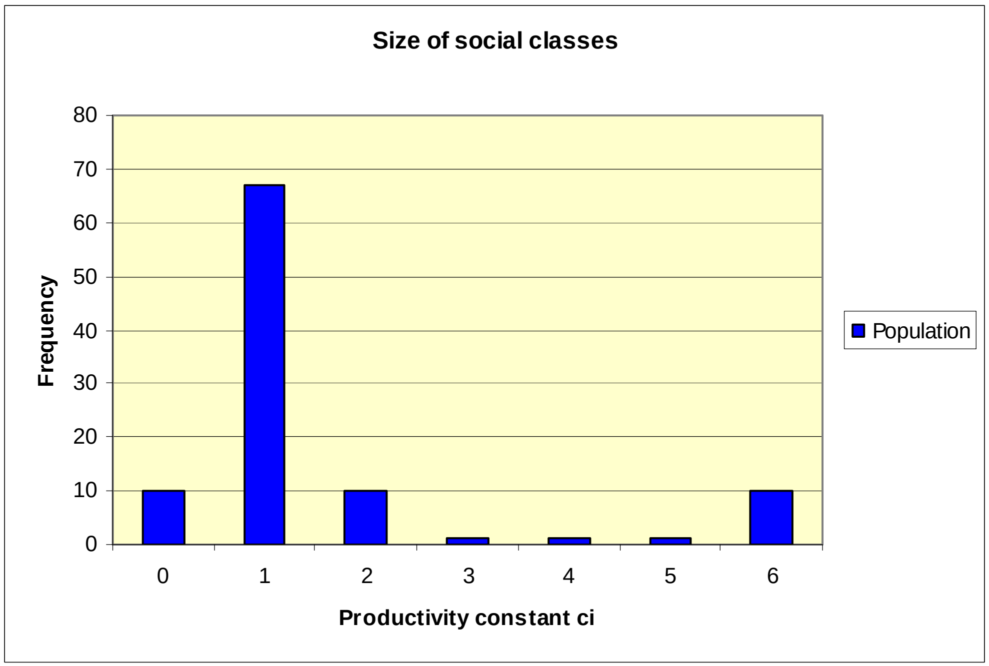


Critical comparison when k = 6

Rawlsian justice requires equality when

$$\frac{p}{q} \leq 0.0269$$

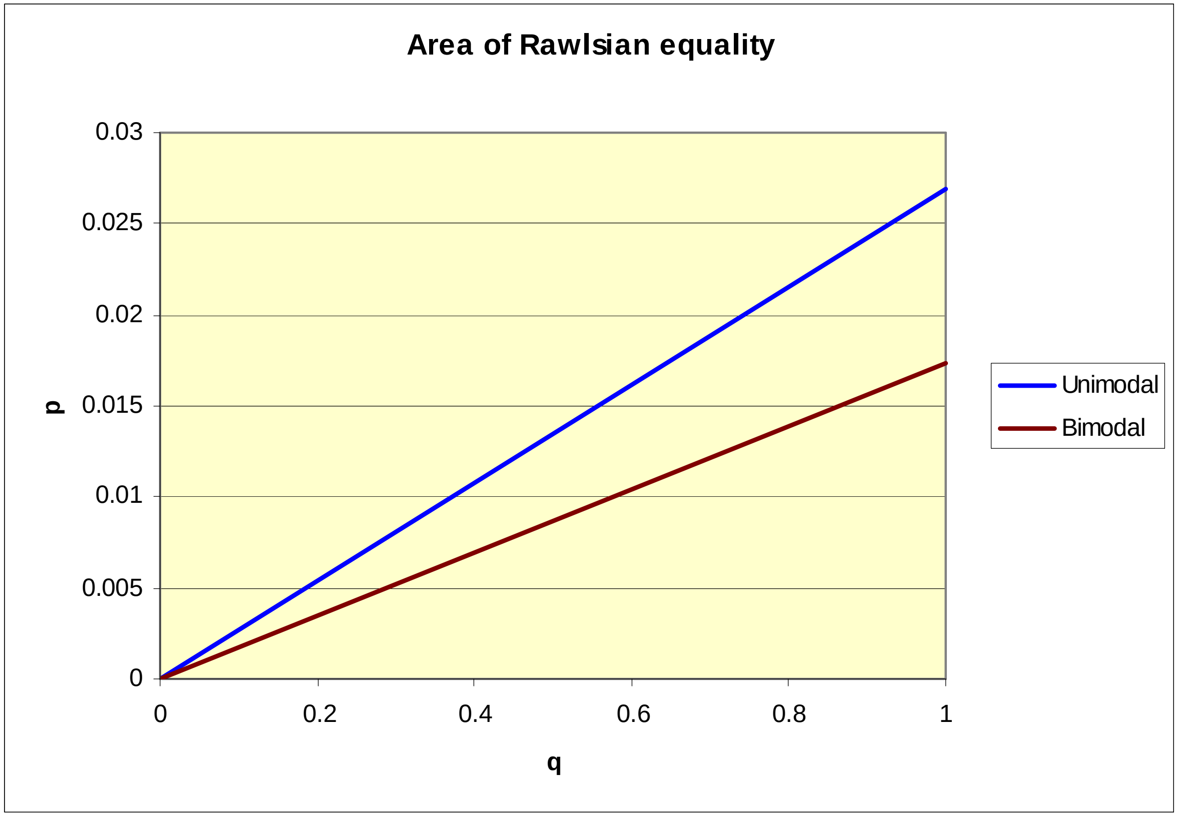
Bimodal productivity distribution



Critical comparison when $k = 6$

Rawlsian justice requires equality when

$$\frac{p}{q} \leq 0.0174$$



Area under each line corresponds to equality.

Equality is more likely in the unimodal distribution.

Rawlsian Model

- In a Rawlsian model:
  - Equality is **more likely** when productivity is insensitive to investment.
  - Equality is **unlikely** when productivity is skewed at the top.
  - Equality is **still possible** when productivity is skewed at the bottom.
  - Equality is **less likely** when people don't care much about getting rich.
  - Equality is **less likely** in a bimodal ("third world") distribution than in a unimodal distribution.