Optimization Models for Equity

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November 2010
Modeling Equity

• There is a growing interest in incorporating equity considerations in mathematical programming models.
  • Not enough to minimize cost or maximize revenue.
  • Also concerned about distribution of resources/benefits.
  • Not obvious how to capture equity in the objective function.
  • Still less obvious how to combine it with an efficiency objective.
Modeling Equity

• Some applications…
  • Single-payer health system.
  • Facility location (e.g., emergency services).
  • Taxation (revenue vs. progressivity).
  • Relief operations.
  • Telecommunications (lexmax, Nash bargaining solution)
Outline

• Today:
  • Utilitarianism
  • Piecewise Linear Modeling
  • Rawlsian Difference Principle
  • Axiomatics
  • Measures of Inequality
  • An Allocation Problem

• Tomorrow:
  • Nash Bargaining Solution
  • Raiffa-Kalai-Smorodinsky Bargaining
  • Disjunctive Modeling
  • Combining Equity and Efficiency
  • Health Care Example
Today’s Outline

• Utilitarianism
  • Utility and production functions
  • The optimization problem
  • Arguments for utilitarianism

• Piecewise Linear Modeling
  • LP model of concave maximization
  • MILP model of nonconcave maximization
Today’s Outline

• Rawlsian Difference Principle
  • The social contract argument
  • The lexmax principle
  • The optimization problem

• Axiomatics
  • Interpersonal comparability
  • Axioms of rational choice
  • Social welfare functions
Today’s Outline

• Measures of Inequality
  • An example
  • Utrilitarian, maximin, and lexmax solution
  • Relative range, max, min
  • Relative mean deviation
  • Variance, coefficient of variation
  • McLoone index
  • Gini coefficient
  • Atkinson index
  • Hoover index
  • Theil index

• An Allocation Problem
Efficiency vs. Equity

- Two classical criteria for distributive justice:
  - Utilitarianism (efficiency)
  - Difference principle of John Rawls (equity)

- These have the must studied philosophical underpinnings.
Utilitarianism

- **Utilitarianism** seeks allocation of resources that maximizes total utility.
  - Let $x_i =$ resources allocated to person $i$.
  - Let $u_i =$ utility enjoyed by person $i$.
  - We have an optimization problem

$$\max \sum_{i} u_i \quad \text{s.t.} \quad u_i = h_i(x_i), \quad \forall i$$

Production functions

Set of feasible resource allocations
Utilitarianism

For example, \( h_i(x_i) = a_i x_i^p \) with different \( a_i \)s for 5 individuals
Utilitarianism

• The individual production function $h_i$ has two components.
  • The value $v_i(x_i)$ created by the individual, as a result of receiving resources $x_i$.
  • The utility $u_i(v_i(x_i)) = h_i(x_i)$ of the value created ($u_i$ is normally concave).
  • So $a_i$ reflects the value function $v_i$ (productivity), and $p$ reflects the combined shape of both functions $v_i$ and $u_i$. 
Utilitarianism

Assume resource distribution is constrained only by a fixed budget. We have the optimization problem

\[
\max \sum_i u_i
\]

\[
u_i = a_i x_i^p, \quad \text{all } i
\]

\[
\sum_i x_i = 1, \quad x_i \geq 0, \quad \text{all } i
\]

This has a closed-form solution

\[
x_i = a_i^{1-p} \left( \sum_{j=1}^n a_j^{1-p} \right)^{-1}
\]
Utilitarianism

Optimal allocations equalize slope (i.e., equal marginal productivity).
Utilitarianism

• Arguments for utilitarianism
  • Can define utility to suit context.
  • Utilitarian distributions incorporate some egalitarian factors:
    • With concave production functions, egalitarian distributions create more utility, ceteris paribus.
    • Inegalitarian distributions create disutility, due to social disharmony.
Utilitarianism

- Egalitarian distributions create more utility?
  - This effect is limited.
  - Utilitarian distributions can be very unequal. Productivity differences are magnified in the allocations.
Utilitarianism

- Egalitarian distributions create more utility?
  - In the example, the **most egalitarian** distribution \((p \to 0)\) assigns resources in proportion to productivity.
Utilitarianism

• Unequal distributions create disutility?
  • Perhaps, but modeling this requires nonseparable utility functions
    \[ u_i = h_i(x_1, \ldots, x_n) \]
    that may result in a problem that is hard to model and solve.
Utilitarianism

- Unequal distributions create disutility?
  - Perhaps, but modeling this requires nonseparable utility functions
    \[ u_i = h_i(x_1, \ldots, x_n) \]
    that may result in a problem that is hard to model and solve.
  - More fundamentally, this defense of utilitarianism is based on contingency, not principle.
Utilitarianism

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  • Perhaps, but modeling this requires nonseparable utility functions
    \[ u_i = h_i(x_1, \ldots, x_n) \]
    that may result in a problem that is hard to model and solve.
  • More fundamentally, this defense of utilitarianism is based on contingency, not principle.
  • If we evaluate the fairness of utilitarian distribution, then there must be another standard of equitable distribution.
  • How do we model the standard we really have in mind?
Modeling Utility

• Ideally, production functions are **concave**, and feasible set is **convex**.
  • For example, $h_i(x_i) = a_i x_i^p$ for $0 < p < 1$ and linear constraints on $x$.
  • Then we solve the problem

$$\max \sum_i h_i(x_i)$$

$$Ax \leq b, \quad x \geq 0$$

by nonlinear programming.

• Any local optimum is a global optimum.
Piecwise Linear Modeling

• Piecewise linear modeling converts **nonlinear programming** to **LP** (linear programming) or **MILP** (mixed integer/linear programming).
  • A key technique.
  • Applies when functions are **separable**.

• Suppose we want to solve

\[
\max \sum_{i} f_i(x_i)
\]
\[
Ax \leq b, \quad x \geq 0
\]
Piecewise Linear Modeling

- If each $f_i$ is **concave**, this reduces (approx.) to an **LP**.
Piecewise Linear Modeling

- If each $f_i$ is **concave**, this reduces (approx.) to an LP.

$$\max \sum_i v_i$$

$$v_i = f_i(a_0) + \sum_j \frac{\Delta f_{ij}}{\Delta a_{ij}} x_{ij}$$

$$x_i = \sum_j x_{ij}$$

$$Ax \leq b, \ x \geq 0$$

where

$$\Delta f_{ij} = f_i(a_{ij}) - f_i(a_{i,j-1})$$

$$\Delta a_{ij} = a_{ij} - a_{i,j-1}$$
• If each $f_i$ is **concave**, this reduces (approx.) to an **LP**.

$$ f_i(x_i) $$

$$ \max \sum_i v_i $$

$$ v_i = f_i(a_0) + \sum_j \frac{\Delta f_{ij}}{\Delta a_{ij}} x_{ij} $$

$$ x_i = \sum_j x_{ij} $$

$$ Ax \leq b, \ x \geq 0 $$

where

$$ \Delta f_{ij} = f_i(a_{ij}) - f_i(a_{i,j-1}) $$

$$ \Delta a_{ij} = a_{ij} - a_{i,j-1} $$

The lower intervals “fill up” first.
Piecewise Linear Modeling

• If each $f_i$ is concave, this reduces (approx.) to an LP.

\[
\begin{align*}
\max & \sum_i v_i \\
v_i &= f_i(a_0) + \sum_j \frac{\Delta f_{ij}}{\Delta a_{ij}} x_{ij} \\
x_i &= \sum_j x_{ij} \\
A x \leq b, & \quad x \geq 0
\end{align*}
\]

where

\[
\begin{align*}
\Delta f_{ij} &= f_i(a_{ij}) - f_i(a_{i,j-1}) \\
\Delta a_{ij} &= a_{ij} - a_{i,j-1}
\end{align*}
\]

The lower intervals “fill up” first.
Piecewise Linear Modeling

- If each $f_i$ is **concave**, this reduces (approx.) to an LP.

\[
\max \sum_i v_i
\]

\[
v_i = f_i(a_0) + \sum_j \frac{\Delta f_{ij}}{\Delta a_{ij}} x_{ij}
\]

\[
x_i = \sum_j x_{ij}
\]

\[
Ax \leq b, \quad x \geq 0
\]

where

\[
\Delta f_{ij} = f_i(a_{ij}) - f_i(a_{i,j-1})
\]

\[
\Delta a_{ij} = a_{ij} - a_{i,j-1}
\]

The lower intervals “fill up” first.
Piecewise Linear Modeling

- If $f_i$ is nonconcave, we can use an MILP model of the piecewise linear approximation.
Piecewise Linear Modeling

• In general, a piecewise linear approximation $v_i$ of $f_i$ has the form

$$x_i a_{i0} b_{i1} a_{i1} b_{i2} a_{ik} b_{ik} x_i$$

The function is continuous when $b_{ij} = a_{i,j+1}$
Piecewise Linear Modeling

- In general, a piecewise linear approximation $v_i$ of $f_i$ has the form

$$x_i = \sum_j \lambda_{ij} f_i(a_{ij}) + \mu_{ij} f_i(b_{ij})$$

$$x_i = \sum_j \lambda_{ij} a_{ij} + \mu_{ij} b_{ij}$$

$$\lambda_{ij} + \mu_{ij} = \delta_{ij}, \text{ all } j$$

$$\sum_j \delta_{ij} = 1$$

$$\lambda_{ij}, \mu_{ij} \geq 0, \quad \delta_{ij} \in \{0,1\}, \text{ all } j$$

The function is continuous when $b_{ij} = a_{i,j+1}$

The best MILP model is:
Piecewise Linear Modeling

- When the piecewise linear function is continuous, **don’t** use the “textbook” model

\[ v_i = \sum_{j=1}^{k+1} \lambda_{ij} f_i(a_{ij}) \]

\[ x_i = \sum_{j=1}^{k+1} \lambda_{ij} a_{ij}, \quad \sum_{j=1}^{k} \lambda_{ij} = 1 \]

\[ \lambda_{ij} \leq \delta_{i,j-1} + \delta_{ij}, \quad j = 2, \ldots, k \]

\[ \lambda_{i1} \leq \delta_{i1}, \quad \lambda_{i,k+1} \leq \delta_{ik}, \quad \sum_{j=1}^{k} \delta_{ij} = 1 \]

\[ \lambda_{ij}, \mu_{ij} \geq 0, \quad \delta_{ij} \in \{0,1\}, \quad j = 1, \ldots, k + 1 \]

where  \[ a_{i,k+1} = b_{ik} \]
Piecewise Linear Modeling

- When the piecewise linear function is continuous, don’t use the “textbook” model

\[
v_i = \sum_{j=1}^{k+1} \lambda_{ij} f_i(a_{ij})
\]

\[
x_i = \sum_{j=1}^{k+1} \lambda_{ij} a_{ij}, \quad \sum_{j=1}^{k} \lambda_{ij} = 1
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\]

\[
\lambda_{ij}, \mu_{ij} \geq 0, \quad \delta_{ij} \in \{0,1\}, \quad j = 1, \ldots, k+1
\]

where \( a_{i,k+1} = b_{ik} \)
Piecewise Linear Modeling

- When the piecewise linear function is continuous, **don’t** use the “textbook” model

\[ v_i = \sum_{j=1}^{k+1} \lambda_{ij} f_i(a_{ij}) \]

\[ x_i = \sum_{j=1}^{k+1} \lambda_{ij} a_{ij}, \quad \sum_{j=1}^{k} \lambda_{ij} = 1 \]

\[ \lambda_{ij} \leq \delta_{i,j-1} + \delta_{ij}, \quad j = 2, \ldots, k \]

\[ \lambda_{i1} \leq \delta_{i1}, \quad \lambda_{i,k+1} \leq \delta_{ik}, \quad \sum_{j=1}^{k} \delta_{ij} = 1 \]

\[ \lambda_{ij}, \mu_{ij} \geq 0, \quad \delta_{ij} \in \{0,1\}, \quad j = 1, \ldots, k + 1 \]

where \( a_{i,k+1} = b_{ik} \)

The “textbook” may tell you to use only the continuous part of the model

\[ v_i = \sum_{j=1}^{k+1} \lambda_{ij} f_i(a_{ij}) \]

\[ x_i = \sum_{j=1}^{k+1} \lambda_{ij} a_{ij} \]

and declare the \( \lambda_{ij} \) SOS2.

This sacrifices the tight relaxation of the next model...
Piecewise Linear Modeling

- The best model of a continuous piecewise $v_i$ is the “incremental” formulation:

\[ v_i = f_i(a_{i1}) + \sum_{j=2}^{k+1} \frac{\Delta f_{ij}}{\Delta a_{ij}} x_{ij} \]

\[ x_i = a_{i1} + \sum_{j=1}^{k} x_{ij} \]

\[ \Delta a_{ij} \delta_{ij} \leq x_{ij} \leq \Delta a_{ij} \delta_{i,j-1}, \quad j = 3, \ldots, k \]

\[ \Delta a_{i2} \delta_{ij} \leq x_{i2} \leq \Delta a_{i2}, \quad 0 \leq x_{i,k+1} \leq \Delta a_{i,k+1} \delta_{ik} \]

\[ \delta_{ij} \in \{0,1\}, \quad j = 2, \ldots, k \]
Problems with Utilitarianism

- A utility maximizing distribution may be unjust.
  - Disabled or nonproductive people may be neglected.
  - Less talented people who work hard may receive meager wage.
  - Not all jobs can be equally productive. Those with less productive jobs may receive fewer resources.
Rawlsian Difference Principle

- Rawls’ **Difference Principle** seeks to maximize the welfare of the worst off.
  - Also known as **maximin** principle.
  - Another formulation: inequality is permissible only to the extent that it is necessary to improve the welfare of those worst off.

\[
\max_i \min u_i \\
\forall i, \ u_i = h_i(x_i), \ \text{all } i \\
\forall x \in S
\]
Rawlsian Difference Principle

• The root idea is that when I make a decision for myself, I make a decision for anyone in similar circumstances.
  • It doesn’t matter who I am.

• Social contract argument
  • I make decisions (formulate a social contract) in an original position, behind a veil of ignorance as to who I am.
  • I must find the decision acceptable after I learn who I am.
  • I cannot rationally assent to a policy that puts me on the bottom, unless I would have been even worse off under alternative policies.
  • So the policy must maximize the welfare of the worst off.
Rawlsian Difference Principle

• Applies only to **basic goods**.
  • Things that people want, no matter what else they want.
  • Salaries, tax burden, medical benefits, etc.
  • For example, salary differentials may satisfy the principle if necessary to make the poorest better off.

• Applies to smallest **groups** for which outcome is predictable.
  • A lottery passes the test even though it doesn’t maximize welfare of worst off – the loser is unpredictable.
  • …unless the lottery participants as a whole are worst off.
Rawlsian Difference Principle

• The difference rule implies a \textit{lexmax} principle.
  – If applied recursively.

• \textbf{Lexmax (lexicographic maximum) principle:}
  – Maximize welfare of least advantaged class…
  – then next-to-least advantaged class…
  – and so forth.
Rawlsian Difference Principle

- There is apparently no practical math programming model for lexmax.
  \[
  \text{lexmax}\ \{u_1,\ldots,u_n\}
  \]
  \[
  u_i = h_i(x_i), \ \text{all } i
  \]
  \[
  x \in S
  \]

- We can solve the problem sequentially (pre-emptive goal programming).
  - Solve the maximin problem.
  - Fix the smallest \( u_i \) to its maximum value.
  - Solve the maximin problem over remaining \( u_j \).
  - Continue to \( u_n \).
The Difference and Lexmax Principles need not result in equality.
  • Consider the example presented earlier…

Rawlsian Difference Principle
Rawlsian Difference Principle

Utilitarian distribution
Rawlsian Difference Principle

Here, lexmax principle results in equality
Utilitarianism

But consider this distribution…

\( p = 0.5 \)
Utilitarianism

Lexmax doesn’t result in equality
Axiomatics

• The economics literature derives social welfare functions from axioms of rational choice.
  • Some axioms are strong and hard to justify.
  • The social welfare function depends on degree of **interpersonal comparability** of utilities.
  • Arrow’s impossibility theorem was the first result, but there are many others.

• **Social welfare function**
  • A function $f(u_1,\ldots,u_n)$ of individual utilities.
  • An optimization model can find a distribution of utility that maximizes social welfare.
Interpersonal Comparability

• Social Preferences
  • Let $u = (u_1,\ldots,u_n)$ be the vector of utilities allocated to individuals.
  • A social welfare function ranks distributions: $u$ is preferable to $u'$ if $f(u) > f(u')$.

• Invariance transformations.
  • These are transformations $\phi$ of utility vectors under which the ranking of distributions does not change.
  • Each $\phi = (\phi_1,\ldots,\phi_n)$, where $\phi_i$ is a transformation of individual utility $u_i$. 
Interpersonal Comparability

• Ordinal noncomparability.
  • Any $\phi = (\phi_1, \ldots, \phi_n)$ with strictly increasing $\phi_i$s is an invariance transformation.

• Ordinal level comparability.
  • Any $\phi = (\phi_1, \ldots, \phi_n)$ with strictly increasing and identical $\phi_i$s is an invariance transformation.
Interpersonal Comparability

• Cardinal nonncomparability.
  • Any $\phi = (\phi_1,\ldots,\phi_n)$ with $\phi_i(u_i) = \alpha_i + \beta_i u_i$ and $\beta_i > 0$ is an invariance transformation.

• Cardinal unit comparability.
  • Any $\phi = (\phi_1,\ldots,\phi_n)$ with $\phi_i(u_i) = \alpha_i + \beta u_i$ and $\beta > 0$ is an invariance transformation.

• Cardinal ratio scale comparability
  • Any $\phi = (\phi_1,\ldots,\phi_n)$ with $\phi_i(u_i) = \beta u_i$ and $\beta > 0$ is an invariance transformation.
Axioms

- Anonymity
  - Social preferences are the same if indices of $u$'s are permuted.
- Strict pareto
  - If $u > u'$, then $u$ is preferred to $u'$.
- Independence of irrelevant alternatives
  - The preference of $u$ over $u'$ depends only on $u$ and $u'$ and not on what other utility vectors are possible.
- Separability of unconcerned individuals
  - Individuals $i$ for which $u_i = u'_i$ don’t affect the ranking of $u$ and $u'$.
Axiomatics

Theorem
Given ordinal level comparability, any social welfare function \( f \) that satisfies the axioms is lexicographically increasing or lexicographically decreasing. So we get a \textbf{lexmax} or \textbf{lexmin} objective.

Theorem
Given cardinal unit comparability, any social welfare function \( f \) that satisfies the axioms has the form \( f(u) = \sum a_i u_i \) for \( a_i \geq 0 \). So we get a \textbf{utilitarian} objective.
Axiomatics

Theorem
Given **cardinal noncomparability**, any social welfare function $f$ that satisfies the axioms (except anonymity and separability) has the form $f(u) = u_i$ for some fixed $i$. So individual $i$ is a **dictator**.

Theorem
Given **cardinal ratio scale comparability**, any social welfare function $f$ that satisfies the axioms has the form $f(u) = \Sigma_i u_i^p/p$. So we get the production function used in the example.
Measures of Inequality

• Assume we wish to **minimize inequality**.
  • We will survey several measures of inequality.
  • They have different strengths and weaknesses.
  • Minimizing inequality may result in less total utility.

• **Pigou-Dalton** condition.
  • One criterion for evaluating an inequality measure.
  • If utility is transferred from one who is worse off to one who is better off, inequality should increase.
Measures of Inequality

• Measures of Inequality
  • An example
  • Utrilitarian, maximin, and lexmax solution
  • Relative range, max, min
  • Relative mean deviation
  • Variance, coefficient of variation
  • McLoone index
  • Gini coefficient
  • Atkinson index
  • Hoover index
  • Theil index

• An Allocation Problem
Example

Production functions for 5 individuals

![Graph showing production functions for 5 individuals](image-url)
Utilitarian

\[ \text{max} \sum_{i} u_{i} \]

LP model: \[ \text{max} \sum_{i=1}^{5} u_{i} \]

\[ u_{i} = a_{i} x_{i}, \quad 0 \leq x_{i} \leq b_{i}, \text{all } i, \quad \sum_{i} x_{i} = B \]

where \((a_{1}, \ldots a_{5}) = (0.5, 0.75, 1, 1.5, 2)\)

\((b_{1}, \ldots, b_{5}) = (20, 25, 30, 35, 40)\)

\(B = 100\)
Utilitarian
Rawlsian

$$\max \left\{ \min_i \{u_i\} \right\}$$

LP model:  
$$\max u_{\min} + \varepsilon \sum_i u_i$$  
Ensures that solution is Pareto optimal

$$u_{\min} \leq u_i, \text{ all } i$$  
$$u_i = a_i x_i, 0 \leq x_i \leq b_i, \text{ all } i, \sum_i x_i = B$$
Rawlsian
Utilitarian
Lexmax

\[ \text{lexmax}\{u_1, \ldots, u_n\} \]

Sequence of LP models, \(k = 1, \ldots, n - 1:\)

\[
\begin{align*}
\max & \quad u_{\min} \\
u_i &= u_i^*, \quad \text{all } i < k \\
u_{\min} &\leq u_i, \quad \text{all } i \geq k \\
u_i &= a_i x_i, \quad 0 \leq x_i \leq b_i, \quad \text{all } i, \quad \sum_i x_i = B
\end{align*}
\]

Re-index for each \(k\) so that \(u_i\) for \(i < k\) were fixed in previous iterations.
Lexmax
Rawlsian
Utilitarian
Relative Range

\[
\frac{u_{\max} - u_{\min}}{\bar{u}}
\]

where \( u_{\max} = \max\{u_i\} \quad u_{\min} = \min\{u_i\} \quad \bar{u} = \frac{1}{n} \sum_{i} u_i \)

**Rationale:**
- Perceived inequality is relative to the best off.
- A distribution should be judged by the position of the worst-off.
- Therefore, minimize gap between top and bottom.

**Problems:**
- Ignores distribution between extremes.
- Violates Pigou-Dalton condition
Equality Measures: Comparison

Relative range: 0 1.30 2.26
Relative Range

\[ \frac{U_{\text{max}} - U_{\text{min}}}{\bar{U}} \]

This is a **fractional linear programming** problem.

Use Charnes-Cooper transformation to an LP. In general,

\[
\begin{align*}
\min & \quad \frac{cx + c_0}{dx + d_0} \\
\text{subject to} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]

becomes

\[
\begin{align*}
\min & \quad cx' + c_0z \\
\text{subject to} & \quad Ax' \geq bz \\
& \quad dx' + d_0z = 1 \\
& \quad x', z \geq 0
\end{align*}
\]

after change of variable \( x = x'/z \) and fixing denominator to 1.
Relative Range

\[
\frac{U_{\text{max}} - U_{\text{min}}}{\bar{U}}
\]

Fractional LP model: 
\[
\min \frac{u_{\text{max}} - u_{\text{min}}}{(1/n)\sum_i u_i}
\]
\[u_{\text{max}} \geq u_i, \quad u_{\text{min}} \leq u_i, \quad \text{all } i\]
\[u_i = a_i x_i, \quad 0 \leq x_i \leq b_i, \quad \text{all } i, \quad \sum_i x_i = B\]

LP model: 
\[
\min u_{\text{max}} - u_{\text{min}}
\]
\[u_{\text{max}} \geq u'_i, \quad u_{\text{min}} \leq u'_i, \quad \text{all } i\]
\[u'_i = a_i x'_i, \quad 0 \leq x'_i \leq b_i z, \quad \text{all } i, \quad \sum_i x'_i = B z\]
\[(1/n)\sum_i u'_i = 1\]
Relative Range

![Graph showing Relative Range with Cumulative Utility, Utility, and Resources lines.](image-url)
Lexmax

![Graph showing cumulative utility, utility, and resources over time. The x-axis represents time (1 to 5), and the y-axis represents values ranging from 0 to 160. The graph includes three lines: one for cumulative utility, one for utility, and one for resources. The cumulative utility line starts at a lower value and increases significantly, while the utility and resources lines remain relatively flat.]
Relative Max

\[ \frac{U_{\text{max}}}{\bar{U}} \]

**Rationale:**
- Perceived inequality is relative to the best off.
- Possible application to salary levels (typical vs. CEO)

**Problems:**
- Ignores distribution below the top.
- Violates Pigou-Dalton condition
Equality Measures: Comparison

Relative range: 0 1.30 2.26
Relative max: 1 1.73 2.38
Relative Max

\[ \frac{U_{\text{max}}}{U} \]

Fractional LP model:

\[
\min \frac{u_{\text{max}}}{(1/n)\sum_{i} u_i} \\
u_{\text{max}} \geq u_i, \text{ all } i \\
u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{ all } i, \ \sum_i x_i = B
\]

LP model:

\[
\min u_{\text{max}} \\
u_{\text{max}} \geq u'_i, \text{ all } i \\
u'_i = a_i x'_i, \ 0 \leq x'_i \leq b_i z, \ \text{ all } i, \ \sum_i x'_i = B z \\
(1/n)\sum_{i} u'_i = 1
\]
Relative Max

![Graph showing cumulative utility, utility, and resources over time. The y-axis represents values ranging from 0 to 160, and the x-axis represents time from 1 to 5. The graph illustrates the growth trends and relationship between cumulative utility, utility, and resources.]
Relative Range

[Graph showing data trends with labels: Cum. Utility, Utility, Resources]
Relative Min

\[
\frac{u_{\text{min}}}{\bar{u}}
\]

**Rationale:**
- Measures adherence to Rawlsian Difference Principle.
- ...relativized to mean

**Problems:**
- Ignores distribution above the bottom.
- Violates Pigou-Dalton condition
Equality Measures: Comparison

Relative range: 0 1.30 2.26
Relative max: 1 1.73 2.38
Relative min: 1 0.43 0.12
Relative Min

\[ \frac{U_{\text{min}}}{\bar{U}} \]

Fractional LP model:
\[
\max \frac{u_{\text{min}}}{(1/n)\sum_i u_i} \\
u_{\text{min}} \leq u_i, \text{ all } i \\
u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \text{ all } i, \ \sum_i x_i = B
\]

LP model:
\[
\max u_{\text{min}} \\
u_{\text{min}} \geq u_i', \text{ all } i \\
u_i' = a_i x_i', \ 0 \leq x_i' \leq b_i z, \text{ all } i, \ \sum_i x_i' = B z \\
(1/n)\sum_i u_i' = 1
\]
Relative Min

![Graph showing Cum. Utility, Utility, and Resources over time.](image-url)
Relative Max
Relative Range
Relative Mean Deviation

\[ \frac{\sum_{i} |u_i - \bar{u}|}{\bar{u}} \]

**Rationale:**
- Perceived inequality is relative to average.
- Entire distribution should be measured.

**Problems:**
- Violates Pigou-Dalton condition
- Insensitive to transfers on the same side of the mean.
- Insensitive to placement of transfers from one side of the mean to the other.
Equality Measures: Comparison

Relative range: 0 1.30 2.26
Rel. mean dev.: 0 0.42 0.72
Relative Mean Deviation

\[
\frac{\sum_i |u_i - \bar{u}|}{\bar{u}}
\]

Fractional LP model:
\[
\max_i \frac{\sum_i (u_i^+ + u_i^-)}{\bar{u}} \\
u_i^+ \geq u_i - \bar{u}, \ u_i^- \geq \bar{u} - u_i, \ \text{all } i \\
\bar{u} = (1/n) \sum_i u_i \\
u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum_i x_i = B
\]

LP model:
\[
\max \sum_i (u_i^+ + u_i^-) \\
u_i^+ \geq u_i' - 1, \ u_i^- \leq u_i' - 1, \ \text{all } i \\
(1/n) \sum_i u_i' = 1 \\
u_i' = a_i x_i', \ 0 \leq x_i' \leq b_i z, \ \text{all } i, \ \sum_i x_i' = Bz
\]
Relative Mean Deviation
Relative Range
Variance

\[(1/ n)\sum (u_i - \bar{u})^2\]

**Rationale:**
- Weight each utility by its distance from the mean.
- Satisfies Pigou-Dalton condition.
- Sensitive to transfers on one side of the mean.
- Sensitive to placement of transfers from one side of the mean to the other.

**Problems:**
- Weighting is arbitrary?
- Variance depends on scaling of utility.
Variance

\[
\frac{1}{n} \sum_{i} (u_i - \bar{u})^2
\]

Convex nonlinear model: \( \min \frac{1}{n} \sum_{i} (u_i - \bar{u})^2 \)

\[\bar{u} = \frac{1}{n} \sum_{i} u_i\]

\[u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum_{i} x_i = B\]
Variance
Relative Mean Deviation

![Graph showing cumulative utility, utility, and resources over time.](image)
Coefficient of Variation

\[
\left( \frac{1}{n} \sum_{i} (u_i - \bar{u})^2 \right)^{1/2} \frac{\bar{u}}{u}
\]

**Rationale:**
- Similar to variance.
- Invariant with respect to scaling of utilities.

**Problems:**
- When minimizing inequality, there is an incentive to reduce average utility.
- Should be minimized only for fixed total utility.
Equality Measures: Comparison

Relative range: 0 1.30 2.26
Rel. mean dev.: 0 0.42 0.72
Coeff. of variation: 0 0.46 0.81
Coefficient of Variation

\[
\left( \frac{1}{n} \sum_{i} (u_i - \bar{u})^2 \right)^{1/2} \geq \frac{1}{\bar{u}} \sum_{i} (1/2) (1/0) (1/2) \min_{i} (1/0) (1/0) \sum_{i} u_i \cdot b \]

Again use change of variable \( u = u'/z \) and fix denominator to 1.

\[
\min \left( \frac{1}{n} \sum_{i} (u_i - \bar{u})^2 \right)^{1/2} \quad \text{becomes} \quad \min \left( \frac{1}{n} \sum_{i} (u'_i - 1)^2 \right)^{1/2} \]

\( Au \geq b \)
\( u \geq 0 \)

Can drop exponent to make problem convex
Coefficient of Variation

\[
\left(\frac{1}{n}\sum_{i}(u_i - \bar{u})^2\right)^{1/2}
\]

\[
\bar{u} = \frac{1}{n}\sum_{i} u_i
\]

Fractional nonlinear model:

\[
\max \left(\frac{1}{n}\sum_{i}(u_i - \bar{u})^2\right)^{1/2}
\]

\[
\bar{u} = \frac{1}{n}\sum_{i} u_i
\]

\[
u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum x_i = B
\]

Convex nonlinear model:

\[
\min (\frac{1}{n}\sum_{i}(u'_i - 1)^2)
\]

\[
(\frac{1}{n}\sum_{i} u'_i = 1)
\]

\[
u'_i = a_i x'_i, \ 0 \leq x'_i \leq b_i z, \ \text{all } i, \ \sum x'_i = B z
\]
Coefficient of Variation
Variance
Relative Mean Deviation
McLoone Index

\[
\frac{1}{2} \sum_{i : u_i < m} u_i \\
\overline{u}
\]

**Rationale:**
- Ratio of average utility below median to overall average.
- No one wants to be “below average.”
- Pushes average up while pushing inequality down.

**Problems:**
- Violates Pigou-Dalton condition.
- Insensitive to upper half.
Equality Measures: Comparison

Relative range: 0 1.30 2.26
Rel. mean dev.: 0 0.42 0.72
Coeff. of variation: 0 0.46 0.81
McLoone: 1 0.54 0.23
McLoone Index

\[(1/2) \sum_{i: u_i < m} u_i \]

Fractional MILP model:

\[\max \frac{\sum v_i}{\sum u_i}\]

Defines median \(m\)

Defines \(v_i = u_i\) if \(u_i\) is below median

Half of utilities are below median

Selects utilities below median

\[m - My_i \leq u_i \leq m + M(1 - y_i), \ \text{all } i\]

\[v_i \leq u_i, v_i \leq My_i, \ \text{all } i\]

\[\sum y_i < n / 2\]

\[u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum x_i = B\]

\[y_i \in \{0,1\}, \ \text{all } i\]
McLoone Index

\[
\frac{(1/2) \sum_{i: u_i < m} u_i}{\bar{u}}
\]

MILP model:
\[ \max \sum_i v_i' \]
\[
m' - M y_i \leq u'_i \leq m' + M (1 - y_i), \quad \text{all } i
\]
\[
v'_i \leq u'_i, \quad v'_i \leq M y_i, \quad \text{all } i
\]
\[
\sum_i y_i < n/2
\]
\[
u'_i = a_i x'_i, \quad 0 \leq x'_i \leq b_i z, \quad \text{all } i, \quad \sum_i x'_i = B z
\]
\[
y_i \in \{0,1\}, \quad \text{all } i
McLoone Index

[Graph showing cumulative utility and resource distribution over time]
Relative Min
Gini Coefficient

\[
\frac{(1/ n^2) \sum_{i,j} |u_i - u_j|}{2\bar{u}}
\]

**Rationale:**
- Relative mean difference between all pairs.
- Takes all differences into account.
- Related to area above cumulative distribution (Lorenz curve).
- Satisfies Pigou-Dalton condition.

**Problems:**
- Insensitive to shape of Lorenz curve, for a given area.
Gini Coefficient

\[
\frac{1}{n^2} \sum_{i,j}^{} \left| u_i - u_j \right| \frac{1}{2 \bar{u}}
\]

Cumulative utility

Gini coeff. = \frac{\text{blue area}}{\text{area of triangle}}

Individuals ordered by increasing utility

Lorenz curve
Equality Measures: Comparison

<table>
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Gini Coefficient

\[
\frac{1}{n^2} \sum_{i,j} |u_i - u_j| \frac{1}{2\bar{u}}
\]

Fractional LP model:

\[
\max \frac{1}{2n^2} \sum_{ij} (u_{ij}^+ + u_{ij}^-)
\]

\[
u_{ij}^+ \geq u_i - u_j, \quad u_{ij}^- \geq u_j - u_i, \quad \text{all } i, j
\]

\[
\bar{u} = \frac{1}{n} \sum u_i
\]

\[
u_i = a_i x_i, \quad 0 \leq x_i \leq b_i, \quad \text{all } i, \quad \sum x_i = B
\]

LP model:

\[
\max (1/2n^2) \sum_{ij} (u_{ij}^+ + u_{ij}^-)
\]

\[
u_{ij}^+ \geq u'_i - u'_j, \quad u_{ij}^- \geq u'_j - u'_i, \quad \text{all } i, j
\]

\[
(1/ n) \sum u'_i = 1
\]

\[
u'_i = a_i x'_i, \quad 0 \leq x'_i \leq b_i z, \quad \text{all } i, \quad \sum x'_i = B z
\]
Gini Coefficient
Coefficient of Variation
Variance
Historical Gini Coefficient, 1945-2010
Atkinson Index

$$\left[1 - \left(\frac{1}{n}\sum_{i} \left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}\right]$$

**Rationale:**
- Best seen as measuring inequality of resources $x_i$.
- Assumes allotment $y$ of resources results in utility $y^p$.
- This is average utility per individual.
Atkinson Index

\[
1 - \left( \frac{1}{n} \sum_{i} \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p}
\]

**Rationale:**
- Best seen as measuring inequality of resources \(x_i\).
- Assumes allotment \(y\) of resources results in utility \(y^p\).
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum_{i} \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

Rationale:

- Best seen as measuring inequality of resources \( x_i \).
- Assumes allotment \( y \) of resources results in utility \( y^p \).
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.
- This is additional resources per individual necessary to sustain inequality.
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum_i \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

**Rationale:**
- \( p \) indicates “importance” of equality.
- Similar to \( L_p \) norm
- \( p = 1 \) means inequality has no importance
- \( p = 0 \) is Rawlsian (measures utility of worst-off individual).

**Problems:**
- Measures utility, not equality.
- Doesn’t evaluate distribution of utility, only of resources.
- \( p \) describes utility curve, not importance of equality.
Equality Measures: Comparison

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Atkinson Index

\[
1 - \left( \frac{1}{n} \sum \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p}
\]

To minimize index, solve fractional problem

\[
\max \sum_i \left( \frac{x_i}{\bar{x}} \right)^p = \frac{\sum x_i^p}{\bar{x}^p}
\]

\[Ax \geq b, \quad x \geq 0\]

After change of variable \( x_i = x_i'/z \), this becomes

\[
\max \sum_i x_i'^p
\]

\[(1/n)\sum x_i' = 1\]

\[Ax' \geq bz, \quad x' \geq 0\]
Atkinson Index

\[ 1 - \left( \frac{1}{n} \sum_i \left( \frac{x_i}{\bar{x}} \right)^p \right)^{1/p} \]

Fractional nonlinear model:

\[ \max \frac{\sum_i x_i^p}{\bar{x}^p} \]
\[ \bar{x} = \frac{1}{n} \sum_i x_i \]
\[ \sum_i x_i = B, \quad x \geq 0 \]

Concave nonlinear model:

\[ \max \sum_i x_i'^p \]
\[ \frac{1}{n} \sum_i x_i' = 1 \]
\[ \sum_i x_i' = Bz, \quad x' \geq 0 \]
Atkinson index
Hoover Index

$\sum |u_i - \bar{u}| \over (1/2) \sum u_i$

**Rationale:**

- Fraction of total utility that must be redistributed to achieve total equality.
- Proportional to maximum vertical distance between Lorenz curve and 45° line.
- Originated in regional studies, population distribution, etc. (1930s).
- Easy to calculate.

**Problems:**

- Less informative than Gini coefficient?
Cumulative utility = Hoover index max vertical distance

\[ \frac{1}{2} \sum_{i} \frac{\sum_{i} (u_{i} - \bar{u})}{\sum_{i} u_{i}} \]

Hoover index = max vertical distance

Total utility = 1

Lorenz curve
### Equality Measures: Comparison

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<td>0.15</td>
<td>0.28</td>
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</table>
Gini Coefficient
Theil Index

\[ \frac{1}{n} \sum_{i} \left( \frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right) \]

Rationale:

- One of a family of entropy measures of inequality.
- Index is zero for complete inequality (maximum entropy).
- Measures nonrandomness of distribution.
- Described as stochastic version of Hoover index.

Problems:

- Motivation unclear.
- A. Sen doesn’t like it.
Equality Measures: Comparison

Relative range: 0 1.30 2.26
Rel. mean dev.: 0 0.42 0.72
Coeff. of variation: 0 0.46 0.81
McLoone: 1 0.54 0.23
Gini: 0 0.26 0.45
Atkinson: 0.06 0 0.06
Hoover: 0 0.15 0.28
Theil: 0 0.27 0.86
Theil Index

\[
(1/n) \sum_i \left( \frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)
\]

Nasty nonconvex model:

\[
\min (1/n) \sum_i \left( \frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)
\]

\[
\bar{u} = (1/n) \sum_i u_i
\]

\[
u_i = a_i x_i, \ 0 \leq x_i \leq b_i, \ \text{all } i, \ \sum_i x_i = B
\]
Theil Index
Hoover Index
Gini Coefficient
Outline

• Today:
  • Nash Bargaining Solution
  • Raiffa-Kalai-Smorodinsky Bargaining
  • Disjunctive Modeling
  • Combining Equity and Efficiency
  • Health Care Example
An Allocation Problem

• From Yaari and Bar-Hillel, 1983.
• 12 grapefruit and 12 avocados are to be divided between Jones and Smith.
• How to divide justly?

Utility provided by one fruit of each kind

<table>
<thead>
<tr>
<th></th>
<th>Jones</th>
<th>Smith</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grapefruit</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>Avocado</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>
An Allocation Problem

The optimization problem:

\[
\text{max } f(u_1, u_2) \\
u_1 = 100 x_{11}, \quad u_2 = 50 x_{12} + 50 x_{22} \\
x_{i1} + x_{i2} = 12, \quad i = 1, 2 \\
x_{ij} \geq 0, \quad \text{all } i, j
\]

where \( u_i \) = utility for person \( i \) (Jones, Smith)  
\( x_{ij} \) = allocation of fruit \( i \) (grapefruit, avocados)  
to person \( j \)
Utilitarian Solution

\[ f(u_1, u_2) = u_1 + u_2 \]

Smith’s utility

Jones’ utility

Optimal solution

(1200, 600)
Rawlsian (maximin) solution

\[ f(u_1, u_2) = \min \{u_1, u_2\} \]
Bargaining Solutions

• A **bargaining solution** is an equilibrium allocation in the sense that none of the parties wish to bargain further.

  • Because all parties are “satisfied” in some sense, the outcome may be viewed as “fair.”

  • Bargaining models have a **default** outcome, which is the result of a failure to reach agreement.

  • The default outcome can be seen as a **starting point**.
Bargaining Solutions

• Several proposals for the default outcome (starting point):

  • **Zero** for everyone. Useful when only the resources being allocated are relevant to fairness of allocation.

  • **Equal split.** Resources (not necessarily utilities) are divided equally. May be regarded as a “fair” **starting point**.

  • **Strongly pareto set.** Each party receives resources that can benefit no one else. Parties can always agree on this.
Nash Bargaining Solution

- The **Nash bargaining solution** maximizes the social welfare function

\[ f(u) = \prod_{i} (u_i - d_i) \]

where \( d \) is the default outcome.

- **Not** the same as **Nash equilibrium**.
- It maximizes the **product of the gains** achieved by the bargainers, relative to the fallback position.
- Assume feasible set is **convex**, so that Nash solution is unique (due to strict concavity of \( f \)).
Nash Bargaining Solution

Nash solution maximizes area of rectangle

Feasible set
Nash Bargaining Solution

Nash solution maximizes area of rectangle.

Feasible set.
Nash Bargaining Solution

Nash solution maximizes area of rectangle

Feasible set
Nash Bargaining Solution

• The optimization problem has a concave objective function if we maximize log $f(u)$.

\[
\max \log \prod_{i} (u_i - d_i) = \sum_{i} \log (u_i - d_i)
\]

$u \in S$

• Problem is relatively easy if feasible set $S$ is convex.
Nash Bargaining Solution

From Zero
Nash Bargaining Solution
From Equality

\[
\begin{align*}
(600, 600) \\
(900, 750)
\end{align*}
\]
Nash Bargaining Solution

- **Strongly pareto set** gives Smith all 12 avocados.
  - Nothing for Jones.
  - Results in utility \((u_1, u_2) = (0, 600)\)

Utility provided by one fruit of each kind

<table>
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<tr>
<td>🍊 100</td>
<td>50</td>
</tr>
<tr>
<td>🥑  0</td>
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Nash Bargaining Solution
From Strongly Pareto Set
Axiomatic Justification

- **Axiom 1.** Invariance under translation and rescaling.
  - If we map \( u_i \rightarrow a_i u_i + b_i \), \( d_i \rightarrow a_i d_i + b_i \),
  then bargaining solution \( u_i^* \rightarrow a_i u_i^* + b_i \).

This is **cardinal noncomparability**.
Axiomatic Justification

- **Axiom 1.** Invariance under translation and rescaling.
  - If we map $u_i \rightarrow a_i u_i + b_i$, $d_i \rightarrow a_i d_i + b_i$,
    then bargaining solution $u_i^* \rightarrow a_i u_i^* + b_i$.

- **Strong assumption** – failed, e.g., by utilitarian welfare function
\textbf{Axiomatic Justification}

- **Axiom 2.** Pareto optimality.
  - Bargaining solution is pareto optimal.

- **Axiom 3.** Symmetry.
  - If all $d$'s are equal and feasible set is symmetric, then all $u_i^*$s are equal in bargaining solution.

\begin{tikzpicture}
  \draw[->] (-1,0) -- (2,0) node[below] {$u_1$};
  \draw[->] (0,-1) -- (0,2) node[left] {$u_2$};
  \filldraw[fill=yellow!30] (0,0) -- (1,0) arc (0:90:1) -- (0,0);
  \filldraw[fill=blue!50] (0,0) circle (0.1) node[below] {$d$};
  \filldraw[fill=blue!50] (1,1) circle (0.1) node[above] {$u^*$};
\end{tikzpicture}
Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow’s axiom.
  - If $u^*$ is a solution with respect to $d$...
Axiomatic Justification

**Axiom 4.** Independence of irrelevant alternatives.
- Not the same as Arrow’s axiom.
- If $u^*$ is a solution with respect to $d$, then it is a solution in a smaller feasible set that contains $u^*$ and $d$. 
• **Axiom 4.** Independence of irrelevant alternatives.
  • Not the same as Arrow’s axiom.
  • If $u^*$ is a solution with respect to $d$, then it is a solution in a smaller feasible set that contains $u^*$ and $d$.
  • This basically says that the solution behaves like an **optimum**.
**Axiomatic Justification**

**Theorem.** Exactly one solution satisfies Axioms 1-4, namely the Nash bargaining solution.

**Proof** (2 dimensions).

First show that the Nash solution satisfies the axioms.

**Axiom 1.** Invariance under transformation. If

\[
\prod_i (u_i^* - d_i) \geq \prod_i (u_i - d_i)
\]

then

\[
\prod_i ((a_i u_i^* + b_i) - (a_id_i + b_i)) \geq \prod_i ((a_i u_i + b_i) - (a_id_i + b_i))
\]
Axiomatic Justification

**Axiom 2.** Pareto optimality. Clear because social welfare function is strictly monotone increasing.

**Axiom 3.** Symmetry. Obvious.

**Axiom 4.** Independence of irrelevant alternatives. Follows from the fact that $u^*$ is an optimum.

Now show that only the Nash solution satisfies the axioms...
Axiomatic Justification

Let $u^*$ be the Nash solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(u_1, u_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution $(1,1)$, by Axiom 1:
Axiomatic Justification

Let \( u^* \) be the Nash solution for a given problem. Then it satisfies the axioms with respect to \( d \). Select a transformation that sends
\[
(u_1, u_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)
\]
The transformed problem has Nash solution \((1,1)\), by Axiom 1:

By Axioms 2 & 3, \((1,1)\) is the **only** bargaining solution in the triangle:
Axiomatic Justification

Let $u^*$ be the Nash solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(u_1, u_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution $(1,1)$, by Axiom 1:

By Axioms 2 & 3, $(1,1)$ is the only bargaining solution in the triangle:

So by Axiom 4, $(1,1)$ is the only bargaining solution in blue set.
Axiomatic Justification

Let $u^*$ be the Nash solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(u_1, u_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution $(1,1)$, by Axiom 1:

So by Axiom 4, $(1,1)$ is the only bargaining solution in blue set.

By Axiom 1, $u^*$ is the only bargaining solution in the original problem.
Axiomatic Justification

- **Problems** with axiomatic justification.
  - Axiom 1 (invariance under transformation) is very strong.
  - Axiom 1 denies *interpersonal comparability*.
  - So how can it reflect moral concerns?

\[ u^* \]
Axiomatic Justification

• **Problems** with axiomatic justification.
  • **Axiom 1** (invariance under transformation) is very strong.
  • Axiom 1 denies *interpersonal comparability*.
  • So how can it reflect moral concerns?

• Most attention has been focused on **Axiom 4** (independence of irrelevant alternatives).
  • Will address this later.
Bargaining Justification

Players 1 and 2 make offers $s$, $t$. 
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Let $p = P(\text{player 2 will reject } s)$, as estimated by player 1.

Bargaining Justification
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Then player 1 will stick with $s$, rather than make a counteroffer, if

$$(1 - p)s_1 + pd_1 \geq t_1$$
Players 1 and 2 make offers \( s, t \).

Let \( p = P(\text{player 2 will reject } s) \), as estimated by player 1.

Then player 1 will stick with \( s \), rather than make a counteroffer, if

\[
(1 - p)s_1 + pd_1 \geq t_1
\]

So player 1 will stick with \( s \) if

\[
p \leq \frac{s_1 - t_1}{s_1 - d_1} = r_1
\]
Bargaining Justification

It is rational for player 1 to make a counteroffer $s'$, rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$

So player 1 will stick with $s$ if

$$p \leq \frac{s_1 - t_1}{s_1 - d_1} = r_1$$
Bargaining Justification

It is rational for player 1 to make a counteroffer \( s' \), rather than player 2, if

\[
\frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2
\]

It is rational for player 2 to make the next counteroffer if

\[
\frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2
\]
Bargaining Justification

It is rational for player 1 to make a counteroffer \( s' \), rather than player 2, if

\[
    r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2
\]

It is rational for player 2 to make the next counteroffer if

\[
    r'_1 = \frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2
\]

But

\[
    \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2}
\]
Bargaining Justification

It is rational for player 1 to make a counteroffer $s'$, rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$

It is rational for player 2 to make the next counteroffer if

$$r'_1 = \frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2$$

But

$$\frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} \iff \frac{t_1 - d_1}{s_1 - d_1} \geq \frac{s_2 - d_2}{t_2 - d_2}$$
So we have \((s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\)

Bargaining Justification

It is rational for player 2 to make the next counteroffer if

\[ r_1' = \frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2} = r_2' \]

But

\[ \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} \leftrightarrow \frac{t_1 - d_1}{s_1 - d_1} \geq \frac{s_2 - d_2}{t_2 - d_2} \]
So we have \[(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\]

Bargaining Justification

It is rational for player 2 to make the next counteroffer if

\[r'_1 = \frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2} = r'_2\]

Similarly

\[\frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2}\]
Bargaining Justification

So we have \((s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\)

It is rational for player 2 to make the next counteroffer if

\[
\frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2} = r_2'
\]

Similarly

\[
\frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2}{t_2 - d_2}
\]

\[
\frac{t_1 - d_1}{s_1' - d_1} \leq \frac{s_2' - d_2}{t_2 - d_2}
\]
So we have \((s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\)

and we have \((t_1 - d_1)(t_2 - d_2) \leq (s'_1 - d_1)(s'_2 - d_2)\)

Similarly
\[
\frac{s'_1 - t_1}{s'_1 - d_1} \geq \frac{t_2 - s'_2}{t_2 - d_2}
\]
\[
\frac{t_1 - d_1}{s'_1 - d_1} \leq \frac{s'_2 - d_2}{t_2 - d_2}
\]
So we have \((s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\)
and we have \((t_1 - d_1)(t_2 - d_2) \leq (s_1' - d_1)(s_2' - d_2)\)

This implies an improvement in the Nash social welfare function.
Bargaining Justification

So we have

\[(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)\]

and we have

\[(t_1 - d_1)(t_2 - d_2) \leq (s'_1 - d_1)(s'_2 - d_2)\]

This implies an improvement in the Nash social welfare function.

Given a minimum distance between offers, continued bargaining converges to Nash solution.
Raiffa-Kalai-Smorodinsky Bargaining Solution

• This approach begins with a critique of the Nash bargaining solution.
Raiffa-Kalai-Smorodinsky Bargaining Solution

• This approach begins with a critique of the Nash bargaining solution.
  • The new Nash solution is worse for player 2 even though the feasible set is larger.
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from $d$ to ideal solution.
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from $d$ to ideal solution.
  - The players receive an equal fraction of their possible utility gains.

\[
\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}
\]
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from \( d \) to ideal solution.
  - Replace Axiom 4 with **Axiom 4' (Monotonicity)**: A larger feasible set with same ideal solution results in a bargaining solution that is better (or no worse) for all players.

\[
\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}
\]
Raiffa-Kalai-Smorodinsky Bargaining Solution

• **Optimization model.**
  • Not an optimization problem over original feasible set (we gave up Axiom 4).
  • But it is an optimization problem (pareto optimality) over the line segment from \(d\) to ideal solution.

\[
\max \sum_i u_i \\
(g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_1 - d_1), \text{ all } i \\
u \in S
\]

\[
\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}
\]
Raiffa-Kalai-Smorodinsky Bargaining Solution

• **Optimization model.**
  • Not an optimization problem over original feasible set (we gave up Axiom 4).
  • But it is an optimization problem (pareto optimality) over the line segment from $d$ to ideal solution.

$$\max \sum_{i} u_i$$

$$(g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_1 - d_1), \text{ all } i$$

$u \in S$
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Optimization model.**
  - Not an optimization problem over original feasible set (we gave up Axiom 4).
  - But it is an optimization problem (pareto optimality) over the line segment from $d$ to ideal solution.

\[
\max \sum_{i} u_i \quad \text{constants}
\]

\[
(g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_1 - d_1), \quad \text{all } i
\]

\[u \in S\]

Linear constraint
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Zero

Diagram showing the Raiffa-Kalai-Smorodinsky bargaining solution with the feasible set and the bargaining solution point (800, 800) marked.
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Equality
Raiffa-Kalai-Smorodinsky Bargaining Solution
From Strong Pareto Set
Axiomatic Justification

- **Axiom 1.** Invariance under transformation.
- **Axiom 2.** Pareto optimality.
- **Axiom 3.** Symmetry.
- **Axiom 4′.** Monotonicity.
Axiomatic Justification

**Theorem.** Exactly one solution satisfies Axioms 1-4′, namely the RKS bargaining solution.

**Proof** (2 dimensions).

Easy to show that RKS solution satisfies the axioms.

Now show that **only** the RKS solution satisfies the axioms.
Axiomatic Justification

Let $u^*$ be the RKS solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(g_1, g_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)$$

The transformed problem has RKS solution $u'$, by Axiom 1:
Axiomatic Justification

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By Axioms 2 & 3, $u'$ is the **only** bargaining solution in the red polygon:
Axiomatic Justification

Let $u^*$ be the RKS solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(g_1,g_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has RKS solution $u'$, by Axiom 1:

By Axioms 2 & 3, $u'$ is the only bargaining solution in the red polygon:

The red polygon lies inside blue set. So by Axiom 4', its bargaining solution is no better than bargaining solution on blue set. So $u'$ is the only bargaining solution on blue set.
Axiomatic Justification

Let $u^*$ be the RKS solution for a given problem. Then it satisfies the axioms with respect to $d$. Select a transformation that sends

$$(g_1, g_2) \rightarrow (1,1), \quad (d_1, d_2) \rightarrow (0,0)$$

The transformed problem has RKS solution $u'$, by Axiom 1:

By Axiom 1, $u^*$ is the only bargaining solution in the original problem.
Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** is still in effect.
  - It denies *interpersonal comparability*.
  - Dropping Axiom 4 sacrifices optimization of a social welfare function.
  - This may not be necessary if Axiom 1 is rejected.
  - Needs modification for > 2 players (more on this shortly).
Resistance to an agreement depends on sacrifice relative to sacrifice under no agreement. Here, player 2 is making a larger relative sacrifice:

\[
\frac{g_1 - s_1}{g_1 - d_1} \leq \frac{g_2 - s_2}{g_2 - d_2}
\]

Minimizing resistance to agreement requires minimizing

\[
\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}
\]
Bargaining Justification

Resistance to an agreement $s$ depends on sacrifice relative to sacrifice under no agreement. Here, player 2 is making a larger relative sacrifice:

\[
\frac{g_1 - s_1}{g_1 - d_1} \leq \frac{g_2 - s_2}{g_2 - d_2}
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Minimizing resistance to agreement requires minimizing

\[
\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}
\]

or equivalently, maximizing

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\min_i \left\{ \frac{s_i - d_i}{g_i - d_i} \right\}
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\]

which is achieved by RKS point.
This is the **Rawlsian social contract** argument applied to **gains** relative to the ideal.

Minimizing resistance to agreement requires minimizing

$$\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}$$

or equivalently, maximizing

$$\min_i \left\{ \frac{s_i - d_i}{g_i - d_i} \right\}$$

which is achieved by RKS point.
Problem with KLS Solution

- However, the RKS solution is Rawlsian only for 2 players.
  - In fact, RKS leads to counterintuitive results for 3 players.

Red triangle is feasible set. RKS point is $d$!
**Problem with KLS Solution**

- However, the RKS solution is Rawlsian only for **2 players**.
  - In fact, KLS leads to counterintuitive results for 3 players.

Red triangle is feasible set. RKS point is \( d \)!

Rawlsian point is \( u \).
Summary

- Rawlsian
- Utilitarian
Summary

The diagram illustrates the concepts of Nash bargaining, Rawlsian, and Utilitarian approaches in a cooperative game setting. The axes represent utilities for player 1 ($u_1$) and player 2 ($u_2$), with a range from 0 to 1200.

- **Nash bargaining** is represented by the red lines connecting the origin (0,0) to the point (600,600), indicating a Pareto efficient but not necessarily fair distribution.
- **Rawlsian** distribution is depicted by the green line, emphasizing equality by focusing on the minimum utilities.
- **Utilitarian** approach is shown by the blue line, aiming to maximize total utility, often resulting in a more equal distribution but not necessarily Pareto efficient.

The diagram highlights the trade-offs and theoretical perspectives in cooperative game theory.
Summary

- Nash bargaining
- Raiffa-Kalai-Smorodinsky bargaining
- Rawlsian
- Utilitarian
Mixed Integer Linear Modeling

• MILP modeling is basically **disjunctive modeling**.

• A problem has an MILP model if and only if it represents a **union of polyhedra** with the same recession cone.

• One can always write an MILP model by expressing the problem as a **disjunction of linear systems** that describe polyhedra with the same recession cone.

• In fact, one can write a **convex hull** (sharp) MILP model in this fashion.
A disjunction of linear systems represents a union of polyhedra.

\[
\min_c \quad cx \\
\bigvee_k (A^k x \geq b^k)
\]
Disjunction of linear systems

A disjunction of linear systems represents a union of polyhedra.

We want a model with a convex hull relaxation (tightest linear relaxation).

\[
\min_k cx \\
\bigvee_k \left( A^k x \geq b^k \right)
\]
Disjunction of linear systems

The closure of the convex hull of

\[ \min_k cx \]
\[ \lor_k (A^k x \geq b^k) \]

...is described by

\[ \min cx \]
\[ A^k x^k \geq b^k y^k, \text{ all } k \]
\[ \sum_k y^k = 1 \]
\[ x = \sum_k x^k \]
\[ 0 \leq y^k \leq 1 \]
Why?

To derive convex hull relaxation of a disjunction...

Write each solution as a convex combination of points in the polyhedron

\[
\begin{align*}
\min & \quad cx \\
A^{k}x^{k} & \geq b^{k}, \text{ all } k \\
\sum_{k} y_{k} & = 1 \\
x & = \sum_{k} y_{k} x^{k} \\
0 & \leq y_{k} \leq 1
\end{align*}
\]
Why?

To derive convex hull relaxation of a disjunction...

\[
\begin{align*}
\min \ & cx \\
A^k x^k & \geq b^k, \text{ all } k \\
\sum_k y_k &= 1 \\
x &= \sum_k y_k x^k \\
0 &\leq y_k \leq 1
\end{align*}
\]

Change of variable

\[
x = y_k \bar{x}^k
\]

Write each solution as a convex combination of points in the polyhedron

Convex hull relaxation (tightest linear relaxation)
MILP Representability

A subset $S$ of $\mathbb{R}^n$ is MILP representable if it is the projection onto $x$ of some MILP constraint set of the form

$$Ax + Bu + Dy \geq b$$

$$x, y \geq 0$$

$$x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ y_k \in \{0,1\}$$
MILP Representability

A subset $S$ of $\mathbb{R}^n$ is MILP representable if it is the projection onto $x$ of some MILP constraint set of the form

$$Ax + Bu + Dy \geq b$$

$$x, y \geq 0$$

$$x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ y_k \in \{0,1\}$$

Theorem. $S \subset \mathbb{R}^n$ is MILP representable if and only if $S$ is the union of finitely many polyhedra having the same recession cone.
Example: Fixed charge function

Minimize a fixed charge function:

\[
\begin{align*}
\min & \quad x_2 \\
\ subject \ to & \quad x_2 \geq \begin{cases} 
0 & \text{if } x_1 = 0 \\
 f + cx_1 & \text{if } x_1 > 0 
\end{cases} \\
& \quad x_1 \geq 0
\end{align*}
\]
Example

Minimize a fixed charge function:

\[
\begin{align*}
\min & \quad x_2 \\
\text{subject to} & \quad x_2 \begin{cases} 
0 & \text{if } x_1 = 0 \\
\geq f + cx_1 & \text{if } x_1 > 0
\end{cases} \\
x_1 & \geq 0
\end{align*}
\]

Feasible set
Example

Minimize a fixed charge function:

\[
\begin{align*}
\min & \quad x_2 \\
x_2 & \geq \begin{cases} 
0 & \text{if } x_1 = 0 \\
f + cx_1 & \text{if } x_1 > 0 
\end{cases} \\
x_1 & \geq 0
\end{align*}
\]

Union of two polyhedra \( P_1, P_2 \)
Example

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\begin{align*}
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x_2 & \geq \begin{cases} 
0 & \text{if } x_1 = 0 \\
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\end{cases} \\
x_1 & \geq 0
\end{align*}
\]

Union of two polyhedra \( P_1, P_2 \)
Example

Minimize a fixed charge function:

\[
\begin{align*}
\min & \quad x_2 \\
x_2 & \geq \begin{cases} 
0 & \text{if } x_1 = 0 \\
 f + cx_1 & \text{if } x_1 > 0
\end{cases} \\
x_1 & \geq 0
\end{align*}
\]

The polyhedra have different recession cones.
Example

Minimize a fixed charge function:

Add an upper bound on $x_1$

$$
\begin{align*}
\min \quad & x_2 \\
\text{subject to} \quad & x_2 \geq \begin{cases} 
0 & \text{if } x_1 = 0 \\
f + cx_1 & \text{if } x_1 > 0
\end{cases}
\end{align*}
$$

$$0 \leq x_1 \leq M$$

The polyhedra have the same recession cone.
Modeling a union of polyhedra

Start with a disjunction of linear systems to represent the union of polyhedra.

The $k$th polyhedron is $\{x \mid A^k x \geq b\}$

Introduce a 0-1 variable $y_k$ that is 1 when $x$ is in polyhedron $k$.

Disaggregate $x$ to create an $x^k$ for each $k$.

\[
\min \ cx \\
\bigvee_k (A^k x \geq b^k)
\]

\[
\min \ cx \\
A^k x^k \geq b^k y_k, \ \text{all } k \\
\sum_k y_k = 1 \\
x = \sum_k x^k \\
y_k \in \{0,1\}
\]
Example

Start with a disjunction of linear systems to represent the union of polyhedra

\[
\begin{align*}
\min & \quad x_2 \\
\left( x_1 = 0 \right) \vee \left( 0 \leq x_1 \leq M \right) \\
\left( x_2 \geq 0 \right) \vee \left( x_2 \geq f + cx_1 \right)
\end{align*}
\]

![Diagram showing the union of polyhedra](image-url)
Example

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min \ x_2$$
$$\left( x_1 = 0 \right) \lor \left( 0 \leq x_1 \leq M \right)$$
$$\left( x_2 \geq 0 \right) \lor \left( x_2 \geq f + cx_1 \right)$$

$$\min \ cx$$
$$x_1^1 = 0, \ x_2^1 \geq 0$$
$$0 \leq x_1^2 \leq My_2, \ -cx_1^2 + x_2^2 \geq fy_2$$
$$y_1 + y_2 = 1, \ y_k \in \{0,1\}$$
$$x = x^1 + x^2$$

Introduce a 0-1 variable $y_k$ that is 1 when $x$ is in polyhedron $k$.

Disaggregate $x$ to create an $x^k$ for each $k$. 
Example

To simplify:
Replace $x_1^2$ with $x_1$.
Replace $x_2^2$ with $x_2$.
Replace $y_2$ with $y$.

This yields
\[ \min x_2 \]
\[ 0 \leq x_1 \leq My \]
\[ x_2 \geq fy + cx_1 \]
\[ y \in \{0, 1\} \]

or
\[ \min fy + cx \]
\[ 0 \leq x \leq My \]
\[ y \in \{0, 1\} \]

“Big M”
Combining Equity and Efficiency

• Utilitarian and Rawlsian distributions seem too extreme in practice.
  – How to combine them?
Combining Equity and Efficiency

• Utilitarian and Rawlsian distributions seem too extreme in practice.
  – How to combine them?

• One proposal:
  – Maximize welfare of worst off (Rawlsian)...
  – …until this requires undue sacrifice from others
  – Seems appropriate in health care allocation.
Combining Equity and Efficiency

• In particular:
  
  – Switch from Rawlsian to utilitarian when inequality exceeds $\Delta$. 
Combining Equity and Efficiency

• In particular:

  – Switch from Rawlsian to utilitarian when inequality exceeds \( \Delta \).
  
  – Build mixed integer programming model.
  – Let \( u_i \) = utility allocated to person \( i \)

• For 2 persons:

  – Maximize \( \min_i \{ u_1, u_2 \} \) (Rawlsian) when \( |u_1 - u_2| \leq \Delta \)
  – Maximize \( u_1 + u_2 \) (utilitarian) when \( |u_1 - u_2| > \Delta \)
Two-person Model

Contours of social welfare function for 2 persons.
Two-person Model

Contours of social welfare function for 2 persons.

Rawlsian region $\min\{u_1, u_2\}$
Two-person Model

Contours of **social welfare function** for 2 persons.

Utilitarian region \[ u_1 + u_2 \]

Rawlsian region \[ \min\{u_1, u_2\} \]
Person 1 is harder to treat. But maximizing person 1’s health requires too much sacrifice from person 2.
Advantages

• Only one parameter $\Delta$
  – Focus for debate.
  – $\Delta$ has intuitive meaning (unlike weights)
  – Examine consequences of different settings for $\Delta$
  – Find least objectionable setting
  – Results in a consistent policy
Social Welfare Function

We want continuous contours…
Social Welfare Function

We want continuous contours…

\[ u_1 + u_2 \]

\[ 2\min\{u_1, u_2\} + \Delta \]

So we use affine transform of Rawlsian criterion
Social Welfare Function

The social welfare problem becomes

\[
\begin{align*}
\max & \quad z \\
\text{s.t.} & \quad z \leq \begin{cases} 
2\min\{u_1, u_2\} + \Delta, & \text{if } |u_1 - u_2| \leq \Delta \\
\min\{u_1, u_2\}, & \text{otherwise}
\end{cases}
\end{align*}
\]

constraints on feasible set
MILP Model

Epigraph is union of 2 polyhedra.
MILP Model

Epigraph is union of 2 polyhedra. Because they have different recession cones, there is no MILP model.

Recession directions \((u_1, u_2, z)\)
MILP Model

Impose constraints \(|u_1 - u_2| \leq M\)
MILP Model

This equalizes recession cones.

Recession directions \((u_1, u_2, z)\)
MILP Model

We have the model...

\[
\text{max } z \\
z \leq 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1, 2 \\
z \leq u_1 + u_2 + \Delta(1 - \delta) \\
u_1 - u_2 \leq M, \quad u_2 - u_1 \leq M \\
u_1, u_2 \geq 0 \\
\delta \in \{0, 1\} \\
\text{constraints on feasible set}
\]
MILP Model

We have the model...

\[
\begin{align*}
\text{max} & \quad z \\
z & \leq 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1, 2 \\
z & \leq u_1 + u_2 + \Delta(1 - \delta) \\
u_1 - u_2 & \leq M, \quad u_2 - u_1 \leq M \\
u_1, u_2 & \geq 0 \\
\delta & \in \{0, 1\}
\end{align*}
\]

This is a \textbf{convex hull} formulation.
Rewrite the 2-person social welfare function as...

\[ \Delta + 2u_{\text{min}} + (u_1 - u_{\text{min}} - \Delta)^+ + (u_2 - u_{\text{min}} - \Delta)^+ \]

\[ \min\{u_1, u_2\} \]

\[ \alpha^+ = \max\{0, \alpha\} \]
Rewrite the 2-person social welfare function as…

$$\Delta + 2u_{\min} + (u_1 - u_{\min} - \Delta)^+ + (u_2 - u_{\min} - \Delta)^+$$

$$\min \{u_1, u_2\}$$

$$\alpha^+ = \max \{0, \alpha\}$$

This can be generalized to \(n\) persons:

$$(n - 1)\Delta + nu_{\min} + \sum_{j=1}^{n} (u_j - u_{\min} - \Delta)^+$$
**n-person Model**

Rewrite the 2-person social welfare function as...

\[
\Delta + 2u_{\min} + (u_1 - u_{\min} - \Delta)^+ + (u_2 - u_{\min} - \Delta)^+ \\
\min\{u_1, u_2\}
\]

\[
\alpha^+ = \max\{0, \alpha\}
\]

This can be generalized to \(n\) persons:

\[
(n - 1)\Delta + nu_{\min} + \sum_{j=1}^{n} (u_j - u_{\min} - \Delta)^+
\]

Epigraph is a union of \(n!\) polyhedra with same recession direction \((u, z) = (1, \ldots, 1, n)\) if we require \(|u_i - u_j| \leq M\)

So there is an MILP model...
To avoid $n!$ 0-1 variables, add auxiliary variables $w_{ij}$

$$
\begin{align*}
& \text{max } z \\
& z \leq u_i + \sum_{j \neq i} w_{ij}, \text{ all } i \\
& w_{ij} \leq \Delta + u_i + \delta_{ij} (M - \Delta), \text{ all } i, j \text{ with } i \neq j \\
& w_{ij} \leq u_j + (1 - \delta_{ij}) \Delta, \text{ all } i, j \text{ with } i \neq j \\
& u_i - u_j \leq M, \text{ all } i, j \\
& u_i \geq 0, \text{ all } i \\
& \delta_{ij} \in \{0,1\}, \text{ all } i, j \text{ with } i \neq j \\
\end{align*}
$$
**n-person MILP Model**

To avoid $n!$ 0-1 variables, add auxiliary variables $w_{ij}$

\[
\begin{align*}
\text{max } & \quad z \\
\text{s.t. } & \quad z \leq u_i + \sum_{j \neq i} w_{ij}, \text{ all } i \\
& \quad w_{ij} \leq \Delta + u_i + \delta_{ij}(M - \Delta), \text{ all } i, j \text{ with } i \neq j \\
& \quad w_{ij} \leq u_j + (1 - \delta_{ij})\Delta, \text{ all } i, j \text{ with } i \neq j \\
& \quad u_i - u_j \leq M, \text{ all } i, j \\
& \quad u_i \geq 0, \text{ all } i \\
& \quad \delta_{ij} \in \{0,1\}, \text{ all } i, j \text{ with } i \neq j
\end{align*}
\]

**Theorem.** The model is correct (not easy to prove).
**$n$-person MILP Model**

To avoid $n!$ 0-1 variables, add auxiliary variables $w_{ij}$

max $z$

$z \leq u_i + \sum_{j \neq i} w_{ij}$, all $i$

$w_{ij} \leq \Delta + u_i + \delta_{ij}(M - \Delta)$, all $i, j$ with $i \neq j$

$w_{ij} \leq u_j + (1 - \delta_{ij})\Delta$, all $i, j$ with $i \neq j$

$u_i - u_j \leq M$, all $i, j$

$u_i \geq 0$, all $i$

$\delta_{ij} \in \{0,1\}$, all $i, j$ with $i \neq j$

**Theorem.** The model is correct (not easy to prove).

**Theorem.** This is a convex hull formulation (not easy to prove).
\textbf{n-group Model}

In practice, funds may be allocated to groups of different sizes. For example, disease/treatment categories.

Let \( \bar{u}_i \) = average utility gained by a person in group \( i \)

\( n_i \) = size of group \( i \)
*n*-group Model

2-person case with $n_1 < n_2$. Contours have slope $-\frac{n_1}{n_2}$
**n-group MILP Model**

Again add auxiliary variables $w_{ij}$

\[
\begin{align*}
\text{max } & \quad z \\
\text{s.t. } & \quad z \leq (n_i - 1)\Delta + n_i \bar{u}_i + \sum_{j \neq i} w_{ij}, \text{ all } i \\
\quad & \quad w_{ij} \leq n_j (\bar{u}_i + \Delta) + \delta_{ij} n_j (M - \Delta), \text{ all } i, j \text{ with } i \neq j \\
\quad & \quad w_{ij} \leq \bar{u}_j + (1 - \delta_{ij}) n_j \Delta, \text{ all } i, j \text{ with } i \neq j \\
\quad & \quad \bar{u}_i - \bar{u}_j \leq M, \text{ all } i, j \\
\quad & \quad \bar{u}_i \geq 0, \text{ all } i \\
\quad & \quad \delta_{ij} \in \{0,1\}, \text{ all } i, j \text{ with } i \neq j
\end{align*}
\]

**Theorem.** The model is correct.

**Theorem.** This is a convex hull formulation.
Health Example

Measure utility in QALYs (quality-adjusted life years).

QALY and cost data based on Briggs & Gray, (2000) etc.

Each group is a disease/treatment pair.

Treatments are discrete, so group funding is all-or-nothing.

Divide groups into relatively homogeneous subgroups.

\[ u_1 \]
Health Example

Add constraints to define feasible set...

\[
\begin{align*}
&\text{max } z \\
&z \leq (n_i - 1)\Delta + n_i\bar{u}_i + \sum_{j \neq i} w_{ij}, \text{ all } i \\
&w_{ij} \leq n_j (\bar{u}_i + \Delta) + \delta_{ij} n_j (M - \Delta), \text{ all } i, j \text{ with } i \neq j \\
&w_{ij} \leq \bar{u}_j + (1 - \delta_{ij}) n_j \Delta, \text{ all } i, j \text{ with } i \neq j \\
&\bar{u}_i - \bar{u}_j \leq M, \text{ all } i, j \\
&\bar{u}_i \geq 0, \text{ all } i \\
&\delta_{ij} \in \{0,1\}, \text{ all } i, j \text{ with } i \neq j \\
&\bar{u}_i = q_i y_i + \alpha_i \\
&\sum_{i} n_i c_i y_i \leq \text{budget} \\
&y_i \in \{0,1\}, \text{ all } i
\end{align*}
\]

\(u_1\)

\(y_i\) indicates whether group \(i\) is funded
<table>
<thead>
<tr>
<th>Intervention</th>
<th>Cost per person $c_i$ (£)</th>
<th>QALYs gained $q_i$</th>
<th>Cost per QALY without intervention $\alpha_i$ (£)</th>
<th>Subgroup size $n_i$</th>
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<tr>
<td>Pacemaker for atroioventricular heart block</td>
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<td>CABG for triple vessel disease</td>
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<td>CABG for double vessel disease</td>
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### QALY & cost data

#### Part 2

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<thead>
<tr>
<th>Intervention</th>
<th>Cost per person ($c_i$)</th>
<th>QALYs gained ($q_i$)</th>
<th>Cost per QALY ($\alpha_i$)</th>
<th>Subgroup size ($n_i$)</th>
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<tr>
<td><strong>Kidney transplant</strong></td>
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<td><strong>Kidney dialysis</strong></td>
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<td><strong>2-5 years survival</strong></td>
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<td><strong>5-10 years survival</strong></td>
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<td><strong>At least 10 years survival</strong></td>
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## Results

Total budget £3 million

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<th>Δ range</th>
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<th>Aortic valve</th>
<th>CABG L</th>
<th>3</th>
<th>2</th>
<th>Heart trans.</th>
<th>Kidney trans.</th>
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## Results

Utilitarian solution

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More dialysis with larger $\Delta$, beginning with longer life span

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### Results

Most rapid change. Possible range for politically acceptable compromise

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## Results

32 groups, 1089 integer variables
Solution time (CPLEX 12.2) is < 0.5 sec for each $\Delta$

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Solution time vs. $\Delta$

![Graph showing solution time vs. Delta with different group numbers and their corresponding time values in seconds.]
Future Work

• Generalize Rawlsian criterion to lexmax.

• Find principled justification for choice of $\Delta$. 