Combining Equity and Utilitarianism in a Mathematical Programming Model

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Abstract

We discuss the problem of combining the conflicting objectives of equity and utilitarianism, for social policy making, in a single mathematical programming model. The definition of equity we use is the Rawlsian one of maximising the minimum utility over individuals or classes of individuals. However, when the disparity of utility becomes too great, the objective becomes progressively utilitarian. Such a model is particularly applicable to health provision although also applicable in other areas. The building of such a model raises technical issues, as the combined objective is not only non-convex, but its epigraph is not MIP representable in its initial form. A method of making it MIP representable is given. We also show that the resultant integer programming model is "sharp" in the sense that its linear programming relaxation describes the convex hull of the feasible set (before extra resource allocation or policy constraints are added).

1 Introduction

The dilemma over whether to pursue policies of increasing equity (sometimes regarded as "fairness") or utilitarianism ("total good") faces all societies. Such policies are often in conflict and have been addressed by a number of authors (e.g., Binmore (2005), Sen and B. Williams (1982)). Should we attempt to reduce differences in wealth at the expense of economic efficiency? Is equity in health provision more important than maximising the aggregate health of the population?

Utilitarianism was advocated by Bentham and Mill in the 18th and 19th centuries; that is, maximising total utility irrespective of differences between individuals (or classes of individuals). Equity (egalitarianism) can be formulated in different ways. In this paper we choose the maximin principle enunciated by Rawls (1972) (although Rawls did not include health in his analysis); that is, one seeks to create utilities so as to maximise the utility of the worst off. There is considerable evidence to suggest that this is considered by the majority of the population to be the most acceptable policy to pursue in health matters (e.g., Yaari and Bar-Hillel (1984)). But most people regard it as unreasonable to take such a policy to its extreme; that is, to continue with such a policy when it takes too many resources from others. There is some (indirect) evidence for this in Williams and Cookson (2007) and Yaari and Bar-Hillel (1984). Hence we switch to a utilitarian objective in extreme circumstances.

Our discussion is perhaps most relevant to health provision but is also relevant in other areas, such as the allocation of educational resources or even to timing traffic lights, given the incompatility between maxising traffic flow and minimising any person's maximum waiting time.

In Section 2 we give a formal statement of the problem. Then, in Section 3 we give a mathematical statement of the problem involving two people (or two classes of people) to create intuition for the n-person case. This problem is then formulated as a mixed integer programme (MIP) in Section 4. It is shown that this formulation is "sharp." Jeroslow[7] defined an MIP formulation to be sharp if the linear programming (LP) relaxation of its feasible set results in the (closure of) the convex hull of feasible integer solutions, making it the "best" possible formulation as a mathematical programme. Of course, this is only the "core" of a practical model. Additional (problem-specific) constraints will be added to impose resource limitations and policy decisions, which will constrain the possible values of the utilities. This will result in a genuine MIP model of the allocation problem, for which integer programming methods will be required. However, the original MIP model (before adding problem-specific constraints) will be the "best" possible in terms of sharpness. We then, in Sections 5 and 6, state and formulate the "many-person" problem as an MIP and show our formulation to be sharp. In Section 7 we give a numerical example, and in Section 8 we suggest possible variants of our model. Finally, in Section 9, we mention other relevant papers concerning both this problem and the technical issues of modelling it.

2 Statement of the Problem

We suppose that a population consists of individuals (or classes of individuals) and that our policies would result in an allocation of utilities $u_1, u_2, ..., u_n$ to these individuals. In the health context these utilities could be QALYs (Quality Adjusted Life Years) (e.g. Broome (1988), Dolan (1998)).

We will endeavour to implement policies (e.g. resource allocation) that will reduce the disparity $|u_i - u_j|$ between any two individuals. However such policies can only be regarded as "reasonable" so long as this disparity is not excessive. Following a suggestion in Williams and Cookson (2000), we will switch from a Rawlsian to a utilitarian criterion when the utility difference exceeds a threshold; that is, when $|u_i - u_j| \geq \Delta$. For example, diversion of resrouces to chronically ill members of the population should not excessively affect the quality of life of fitter individuals who receive less medical care as a result.

The level at which to set Δ is clearly judgemental and likely to be a point of disagreement among the parties concerned. However, once a value for Δ has been settled upon, the MIP model allows the same policy to be applied consistently whenever a budgeting decision is taken. It is necessary to agree on an efficiency/equity compromise only once, when the value of Δ is selected, rather than revisiting the issue every time it comes up in practice.

Furthermore, the MIP model enables policy makers to examine the consequences of a given value of Δ across a wide variety of cases. They can compute allocations in typical scenarios for each of several values of Δ. Stakeholders can then examine each scenario and indicate which allocation they prefer. The value of Δ that results in the most popular (or least objectionable) allocation might then be selected. Once it is selected, the stakeholders can be assured that the same policy is applied consistently across the board.

3 The Two-person Problem

For the two person case the situation is easy to illustrate. The objective contours (isoquants) are shown in Figure 1. The L-shaped portions of the isoquants, whose straight segments are of length Δ, represent the equitable (maximin) objective $\min\{u_1, u_2\}$. The diagonal portions represent the utilitarian objective $u_1 + u_2$. The relevance of the quantity M is discussed below.

Two individuals are to be allotted utilities u_1, u_2 respectively. When $|u_1 - u_2| \geq \Delta$, the aggregate utility function is utilitarian and given by $u_1 + u_2$. Otherwise the utility function is based on $\min\{u_1, u_2\}$. Since we want the combined utility function to be continuous, we define the utility in the latter case to be $2\min\{u_1, u_2\} + \Delta$. The optimization problem is therefore

$$
\begin{aligned}
&\max z \\
&z \leq \begin{cases} 2\min\{u_1, u_2\} + \Delta & \text{if } |u_1 - u_2| \leq \Delta \\ u_1 + u_2 & \text{otherwise} \end{cases} \\
&u_1, u_2 \geq 0
\end{aligned}
\tag{1}
$$

Additional constraints can be placed on (u_1, u_2) to represent resource limitations.

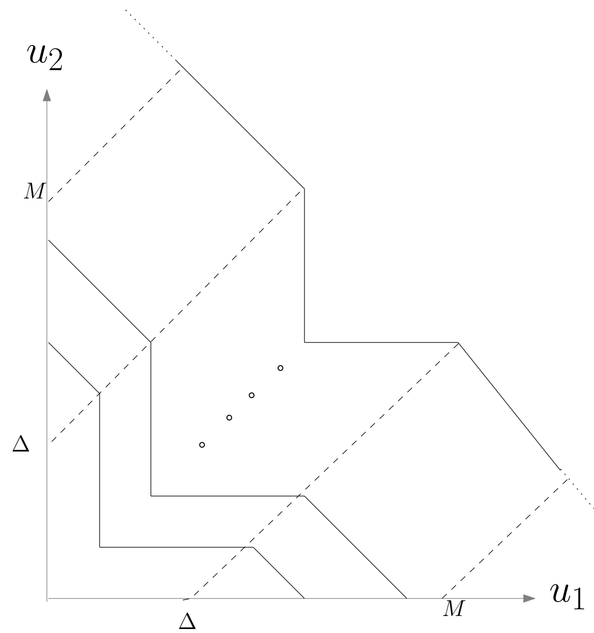

Figure 1: Isoquants for the 2-Person Case, from Williams and Cookson (2000).

4 Mixed Integer Formulation

In order for a problem to be "MIP representable" (Jeroslow, 1987, 1989), its epigraph must be representable as the union of a finite number of polyhedra with the same recession directions. We do not repeat the definitions of a epigraph and recession directions here but refer the reader to Jeroslow (1987, 1989), Hooker (2009), or Williams (2009). If the polyhedra do not have the same recession directions, then some innocuous constraints need to be added to equalise the recession cones.

The epigraph of (1) is the union of two polyhedra, defined respectively by the two disjuncts:

$$\begin{pmatrix} z \leq 2u_1 + \Delta \\ z \leq 2u_2 + \Delta \\ u_1, u_2 \geq 0 \end{pmatrix} \vee \begin{pmatrix} z \leq u_1 + u_2 \\ u_1, u_2 \geq 0 \end{pmatrix}$$

The first disjunct corresponds to the maximin case and the second to the utilitarian case.

The two polyhedra have different recession cones. The recession cone for the first is spanned by the four vectors

$$(u_1, u_2, z) = (1, 1, 2), (1, 0, 0), (0, 1, 0), (0, 0, -1)$$

The recession cone for the second is spanned by the vectors

$$(u_1, u_2, z) = (1, 1, 0), (1, 0, 1), (0, 0, -1)$$

However, if we add the following constraints to each disjunct:

$$\begin{aligned} u_1 - u_2 &\le M \\ u_2 - u_1 &\le M \end{aligned}$$

then the polyhedra have the same recession cone, spanned by the vectors

$$(u_1, u_2, z) = (1, 1, 2), (0, 0, -1)$$

The epigraph is now represented by the big-M model

$$\begin{array}{ll} z \le 2u_1 + \Delta + (M - \Delta)\delta & (a) \\ z \le 2u_2 + \Delta + (M - \Delta)\delta & (b) \\ z \le u_1 + u_2 + \Delta(1 - \delta) & (c) \\ u_1 - u_2 \le M & (d) \\ u_2 - u_1 \le M & (e) \\ u_1, u_2 \ge 0, \;\; 0 \le \delta \le 1 & \\ \delta \text{ integral} & \end{array} \tag{2}$$

We can also give the two polyhedra the same recession cone (namely, the origin) by imposing bounds $u_1, u_2 \le M$. In this case the formulation is the same except that constraints (d) and (e) are replaced by $u_1, u_2 \le M$.

5 Convex Hull

We can show that (2) is a sharp formulation.

Theorem 1 *The system (2) without the integrality constraint describes the convex hull of the epigraph of (1).*

Proof. We use a dual proof. It suffices to show that any inequality

$$z \le a_1 u_1 + a_2 u_2 + b \tag{3}$$

that is valid for (1) is implied by (2). For this it suffices to show that (3) is dominated by a surrogate (nonnegative linear combination) of the system (2). We first combine (2a) and (2c) using multipliers $1/(M - \Delta)$ and $1/\Delta$, respectively, to eliminate δ and obtain the first inequality below. We similarly combine (2b) and (2c) to obtain the second inequality below. The system (2) therefore implies

$$\begin{aligned}
& z \le \left(1+\frac{\Delta}{M}\right)u_1 + \left(1-\frac{\Delta}{M}\right)u_2 + \Delta && (\alpha) \\
& z \le \left(1-\frac{\Delta}{M}\right)u_1 + \left(1+\frac{\Delta}{M}\right)u_2 + \Delta && (1-\alpha) \\
& u_1 - u_2 \le M && (\beta) \\
& u_2 - u_1 \le M && (\beta') \\
& u_1, u_2 \ge 0, \quad 0 \le \delta \le 1
\end{aligned} \tag{4}$$

We will show that (3) is dominated by a surrogate of (4).

First we observe that $(u_1, u_2, z) = (0, 0, \Delta)$ is feasible in (1) and must therefore satisfy (3). Substituting these values into (3), we obtain $b \ge \Delta$. Also, for any $t \ge 0$,

$$(u_1, u_2, z) = (t, t, 2t + \Delta)$$

is feasible in (1), which implies

$$a_1 + a_2 \ge 2 - \frac{b - \Delta}{t}$$

Letting $t \to \infty$, we get that $a_1 + a_2 \ge 2$. It suffices to show that any (3) with $a_1 + a_2 = 2$ is dominated by a surrogate of (4), because in this case (3) with $a_1 + a_2 > 2$ can be obtained by adding multiples of $u_1 \ge 0$ and/or $u_2 \ge 0$ to an inequality with $a_1 + a_2 = 2$.

We now associate multipliers with the system (4) as shown and consider three cases.

Case 1: $1 - (\Delta/M) \le a_1 \le 1 + (\Delta/M)$. If we let

$$\alpha = \tfrac{1}{2}\left(\frac{M}{\Delta}(a_1 - 1) + 1\right)$$

and $\beta = \beta' = 0$, then the multipliers α and $1 - \alpha \ge 0$ are nonnegative, and the linear combination is

$$z \le a_1 u_1 + a_2 u_2 + \Delta \tag{5}$$

where $a_2 = 2 - a_1$. But because $b \ge \Delta$, (3) is dominated by (5) and therefore by a surrogate of (4).

Case 2: $a_1 < 1 - (\Delta/M)$. Let $\alpha = 1/2$, $\beta = 0$, and $\beta' = 1 - a_1$. Then $\beta' \ge 0$ follows from $a_1 < 1 - (\Delta/M)$, and the the linear combination is again (5) with $a_2 = 2 - a_1$. As before, (5) dominates (3).

Case 3: $a_1 > 1 + (\Delta/M)$. This is similar to Case 2, and the proof is complete.

6 Many-person Problem

We now allot utilities $u_1, \ldots, u_n$ to n individuals. One way to generalize the two-person problem (1) is to observe that (1) can be written

$$\begin{aligned}
&\max z \\
&z \leq \Delta + 2u_{\min} + \max\{0, u_1 - u_{\min} - \Delta\} + \max\{0, u_2 - u_{\min} - \Delta\} \\
&u_1, u_2 \geq 0
\end{aligned} \tag{6}$$

where $u_{\min} = \min\{u_1, u_2\}$. Thus each person i makes a utilitarian contribution if u_i differs from $u_{\min}$ more than Δ. If $u_1 > u_2 + \Delta$, the first max term of (6) contributes $u_1 - u_{\min} - \Delta$ and the second max term nothing, yielding $u_1 + u_2$ altogether, and similarly if $u_2 > u_1 + \Delta$. Otherwise, both max terms vanish.

The pattern in (6) can be generalized as follows:

$$\begin{aligned}
&\max z \\
&z \leq (n-1)\Delta + nu_{\min} + \sum_{j=1}^{n} \max\{0, u_j - u_{\min} - \Delta\} \\
&u_i \geq 0
\end{aligned} \tag{7}$$

where $u_{\min} = \min_i\{u_i\}$. Thus person j contributes u_j if u_j differs from $u_{\min}$ more than Δ and is otherwise represented by $u_{\min}$. If everyone makes a utilitarian contribution, then the summation in (7) becomes

$$-(n-1)\Delta - (n-1)u_k + \sum_{i \neq k} u_i$$

where $u_{\min} = u_k$, and the inequality constraint in (7) becomes $z \leq \sum_{i=1}^{n} u_i$.

7 Mixed Integer Representability

As in the two-person case, the n-person problem can be formulated as an MIP if we suppose that $|u_i - u_j| \leq M$ for all i, j. To see this, we can view the epigraph as a union of polyhedra as follows. For each permutation $\pi = (\pi_1, \ldots, \pi_n)$ of $1, \ldots, n$ and each integer $k \in \{1, \ldots, n\}$, let the polyhedron $P_{\pi k}$ be defined by

$$\begin{aligned}
&z \leq (n-1)\Delta + nu_{\pi_1} + \sum_{j=k+1}^{n} (u_{\pi_j} - u_{\pi_1} - \Delta) \\
&u_{\pi_i} \leq u_{\pi_{i+1}}, \quad i = 1, \ldots, n-1 \\
&u_i - u_j \leq M, \quad u_i \geq 0, \text{ all } i, j \text{ with } i \neq j
\end{aligned} \tag{8}$$

Lemma 2 *The feasible set of (7) is identical to the union of polyhedra $P_{\pi k}$ over all permutations π of $1,\ldots,n$ and all $k\in\{1,\ldots,n\}$.*

Proof. We first show that every feasible solution (u,z) of (7) belongs to some $P_{\pi k}$. Suppose without loss of generality that $u_1\le\cdots\le u_n$. Let $(\pi_1,\ldots,\pi_n)=(1,\ldots,n)$, and let k be the smallest integer such that $u_k-u_1\ge\Delta$ (or let $k=n$ if $u_k-u_1<\Delta$ for all k). Then the first inequality in (7) becomes

$$z\le(n-1)\Delta+nu_1+\sum_{j=k+1}^{n}(u_j-u_1-\Delta)$$

and therefore $(u,z)\in P_{\pi k}$.

Now we show that any point (u,z) in any given polyhedron $P_{\pi k}$ satisfies (7) and is therefore feasible. Suppose without loss of generality that $\pi=(1,\ldots,n)$, so that $u_1\le\cdots\le u_n$ and $u_j-u_1\ge\Delta$ for $j=k+1,\ldots,n$. Then because $u_{\min}=u_1$, (u,z) satisfies (7) and is therefore feasible.

The recession cone is the same for each polyhedron $P_{\pi k}$, namely the one spanned by the vectors

$$(u,z)=(1,\ldots,1,n),(0,\ldots,0,-1)$$

The feasible set of (7) is therefore mixed integer representable.

8 Mixed Integer Formulation

We can in principle write an MIP model for (7) based on the union of polyhedra $P_{\pi k}$, but a much more compact model is the following:

$$\begin{array}{lr}\max z & \\ z\le u_i+\sum_{j\ne i}w_{ij},\ \text{all } i & (a)\\ w_{ij}\le\Delta+u_i+\delta_{ij}(M-\Delta),\ \text{all } i,j \text{ with } i\ne j & (b)\\ w_{ij}\le u_j+(1-\delta_{ij})\Delta,\ \text{all } i,j \text{ with } i\ne j & (c)\\ u_i-u_j\le M,\ \text{all } i,j \text{ with } i\ne j & (d)\\ u_i\ge 0,\ \text{all } i & (e)\\ 0\le\delta_{ij}\le 1,\ \delta_{ij}\ \text{integer},\ \text{all } i,j \text{ with } i\ne j & (f)\end{array}\qquad(9)$$

An interpretation of the 0-1 variables δ_{ij} is that $\delta_{ij}=1$ when $u_j-u_i\ge\Delta$.

Theorem 3 *Problem (9) is a correct formulation of (7).*

Proof. We must show that any feasible solution of (7) is a solution of (9), and vice-versa. To show the former, consider any feasible solution (u,z) of (7). We show that there are values of w, δ such that (u, z, w, δ) is a feasible solution of (9). We suppose without loss of generality that $u_1 \le \cdots \le u_n$. Set $\delta_{ij} = 1$ when $u_j - u_i \ge \Delta$ and $\delta_{ij} = 0$ otherwise. Also set

$$w_{ij} = \begin{cases} \Delta + \min\{u_i, u_j\} & \text{if } \delta_{ij} = 0 \\ u_j & \text{if } \delta_{ij} = 1 \end{cases} \tag{10}$$

We now show that this solution satisfies (9). Constraints (9d)-(9f) are satisfied by hypothesis. Also (10) satisfies (9b)-(9c) when $\delta_{ij} = 0$ because it sets w_{ij} to the smaller bound, and it satisfies (9b)-(9c) when $\delta_{ij} = 1$ because, in this case, (9c) provides the tighter bound. Finally, to show that (9a) is satisfied it suffices to show

$$z \le u_i + \sum_{\substack{j \\ \delta_{ij} = 0}} (\Delta + \min\{u_i, u_j\}) + \sum_{\substack{j \\ \delta_{ij} = 1}} u_j \tag{11}$$

But this can be written

$$z \le u_i + \sum_{\substack{j \\ \delta_{ij} = 0}} \left(u_i + \Delta + \min\left\{ \begin{matrix} 0, \\ u_j - u_i \end{matrix} \right\} \right) + \sum_{\substack{j \\ \delta_{ij} = 1}} \left(u_i + \Delta + \max\left\{ \begin{matrix} 0, \\ u_j - u_i - \Delta \end{matrix} \right\} \right) \tag{12}$$

because $u_j - u_i \le 0$ when $\delta_{ij} = 0$ and $u_j - u_i - \Delta \ge 0$ when $\delta_{ij} = 1$. We now show that (12) follows from (7). Note first that (7) can be written

$$z \le u_1 + \sum_{\substack{j > 1 \\ \delta_{ij} = 0}} \left(u_1 + \Delta + \max\left\{ \begin{matrix} 0, \\ u_j - u_1 - \Delta \end{matrix} \right\} \right) + \sum_{\substack{j \\ \delta_{ij} = 1}} \left(u_1 + \Delta + \max\left\{ \begin{matrix} 0, \\ u_j - u_1 - \Delta \end{matrix} \right\} \right) \tag{13}$$

because the max in the first summation is equal to its first argument ($\delta_{ij} = 0$ implies $u_j - u_1 - \Delta \le 0$) and the max in the second summation is equal to its second argument (since $\delta_{ij} = 1$). But (13) can be written

$$z \le u_1 + \sum_{\substack{j > 1 \\ \delta_{ij} = 0}} \left(u_i + \Delta + \max\left\{ \begin{matrix} u_1 - u_i, \\ u_j - u_i - \Delta \end{matrix} \right\} \right) + \sum_{\substack{j \\ \delta_{ij} = 1}} \left(u_i + \Delta + \max\left\{ \begin{matrix} u_1 - u_i, \\ u_j - u_i - \Delta \end{matrix} \right\} \right) \tag{14}$$

We can show that (14) implies (12) by showing that the right-hand side of (14) is less than or equal to the right-hand side of (12). First, $u_1 \le u_i$. Now, looking at each term j of the first summation (for $j > 1$), both arguments of the max in (14) are less than or equal both arguments of the min in (12). Thus the max in (14) is less than or equal to the min in (12). The term corresponding to $j = 1$ does not appear in (14), but the corresponding term of (12) is nonnegative. Looking at the second summation in (12) and (14), we see that the max in (14) is equal to the max in (12) because the second argument of the max is the larger one in both (12) and (14).

It remains to show that given any feasible solution (u, z, w, δ) of (9), (u, z) satisfies (7). Suppose without loss of generality that $u_1 \le \cdots \le u_n$. Then letting $i = 1$ in (9a), (9a)–(9c) imply

$$z \le u_1 + \sum_{\substack{\delta_{1j}=1 \\ j>1}} u_j + \sum_{\substack{\delta_{1j}=0 \\ j>1}} (\Delta + \min\{u_1, u_j\}) \tag{15}$$

The first summation reflects the tighter of the two bounds (9b)–(9c) on w_{1j} when $\delta_{1j} = 1$, and the second summation reflects that fact that (9c) provides the tighter bound when $\delta_{1j} = 0$. Now because $u_1 \le u_j$, (15) implies

$$z \le u_1 + \sum_{\substack{\delta_{1j}=1 \\ j>1}} u_j + \sum_{\substack{\delta_{1j}=0 \\ j>1}} (\Delta + u_1) \tag{16}$$

This implies

$$z \le u_1 + \sum_{\substack{\delta_{1j}=1 \\ j>1}} \left(u_j + \max\left\{ \begin{matrix} u_1 - u_j + \Delta, \\ 0 \end{matrix} \right\} \right) + \sum_{\substack{\delta_{1j}=0 \\ j>1}} \left(\Delta + u_1 + \max\left\{ \begin{matrix} 0, \\ u_j - u_1 - \Delta \end{matrix} \right\} \right) \tag{17}$$

because all the max expressions in (17) are nonnegative. But (17) is equivalent to

$$z \le u_1 + \sum_{\substack{\delta_{1j}=1 \\ j>1}} \left(u_1 + \Delta + \max\left\{ \begin{matrix} 0, \\ u_j - u_1 - \Delta \end{matrix} \right\} \right) + \sum_{\substack{\delta_{1j}=0 \\ j>1}} \left(\Delta + u_1 + \max\left\{ \begin{matrix} 0, \\ u_j - u_1 - \Delta \end{matrix} \right\} \right) \tag{18}$$

as is seen by adding and subtracting $u_1 - u_j + \Delta$ in each term of the first summation. Collecting terms, (18) is equivalent to

$$z \le (n-1)\Delta + nu_1 + \sum_{j>1} \max\left\{ \begin{matrix} 0, \\ u_j - u_1 - \Delta \end{matrix} \right\}$$

which implies (7) because $u_1 = u_{\min}$. This shows that (u, z) satisfies (7), as claimed.

The constraints $\delta_{ij} + \delta_{ji} \le 1$ (for $i \ne j$) can be added to (9), but they are redundant. One way of understanding this is to note that only δ_{1j}s are needed to derive (7) from the MILP model (9) in the above proof.

In practice, the model (9) can have alternate optimal solutions with the same value of z but different total utilities. We break ties in favor of greater utility by changing the objective function to $z + \epsilon \cdot \sum_i u_i$ for a very small $\epsilon > 0$.

9 Convex Hull

We can show that (9) is a sharp formulation of (7).

Theorem 4 *The system (9) without integrality constraints describes the convex hull of the epigraph of (7).*

Proof. We use a dual proof. We showed above that any feasible solution of (7) is the projection onto (u, z) of some solution of (9). Given this, it suffices to show that any inequality $z \leq au + b$ that is valid for (7) is implied by (9). For this it is enough to show that $z \leq au + b$ is dominated by a surrogate (nonnegative linear combination) of the system (9). We first combine (9b) and (9c) for each pair i, j using multipliers $1/(M - \Delta)$ and $1/\Delta$, respectively to eliminate δ_{ij} and obtain

$$w_{ij} \leq \frac{\Delta}{M} u_i + \left(1 - \frac{\Delta}{M}\right) u_j, \text{ all } i, j \text{ with } i \neq j \tag{19}$$

Combining (9a) with (19), the system (2) implies

$$\begin{aligned}
&\max z \\
&z \leq \left(1 + (n-1)\frac{\Delta}{M}\right) u_i + \left(1 - \frac{\Delta}{M}\right) \sum_{j \neq i} u_j + (n-1)\Delta, \text{ all } i && (\alpha_i) \\
&u_j - u_i \leq M, \text{ all } i, j && (\beta_{ij}) \\
&u_i \geq 0, \text{ all } i
\end{aligned} \tag{20}$$

We will show that $z \leq au + b$ is dominated by a surrogate of (20).

First we observe that $(u_1, \ldots, u_n, z) = (0, \ldots, 0, (n-1)\Delta)$ is feasible in (7) and must therefore satisfy $z \leq au + b$. Substituting these values into $z \leq au + b$, we obtain $b \geq (n-1)\Delta$. Also, for any $t \geq 0$,

$$(u_1, \ldots, u_n, z) = (t, \ldots, t, nt + (n-1)\Delta)$$

is feasible in (7), which implies

$$\sum_i a_i \geq n - \frac{b - (n-1)\Delta}{t}$$

Letting $t \to \infty$, we get that $\sum_i a_i \geq n$. It suffices to show that any $z \leq au + b$ with $\sum_i a_i = n$ is dominated by a surrogate of (20), because in this case an inequality with $\sum_i a_i > n$ can be obtained by adding multiples of $u_i \geq 0$ to an inequality with $\sum_i a_i = n$.

We let $N = \{1, \ldots, n\}$ and define index sets as follows:

$$I = \left\{ i \in N \;\middle|\; 1 - \frac{\Delta}{M} \leq a_i \leq 1 \right\}, \quad J = \left\{ i \in N \;\middle|\; a_i < 1 - \frac{\Delta}{M} \right\}, \quad K = N \setminus (I \cup J)$$

We next associate multipliers with (20) as shown and define them as follows:

$$\begin{aligned}
\alpha_i &= \begin{cases} \dfrac{M}{n\Delta}\left(a_i - 1 + \dfrac{\Delta}{M}\right) & \text{if } i \in I \\ \dfrac{1 - \alpha(I)}{n - |I|} & \text{otherwise} \end{cases} \\
\beta_{ij} &= \begin{cases} \dfrac{1}{|K|}\left(\dfrac{n - a(I)}{n - |I|} - a_i\right) & \text{if } i \in J,\, j \in K \\ f_{ij} & \text{if } i, j \in K \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{21}$$

where $\alpha(I) = \sum_{j \in I} \alpha_j$, and similarly for $a(I)$ and $a(K)$. The quantities f_{ij} are feasible nonnegative flows on edges (i, j) of a complete directed graph whose vertices correspond to indices in K, with a net supply of $a_i - a(K)/|K|$ at each vertex i. Such flows exist because the net supply over all vertices is $\sum_i (a_i - a(K)/|K|) = 0$.

We first show that the linear combination of constraints in (20) using these multipliers is $z \leq au + (n-1)\Delta$, given that $\sum_i a_i = n$. It is easily checked that $\sum_i \alpha_i = 1$, so that the linear combination has the form $z \leq du + (n-1)\Delta$. It remains to show that $d = a$. We have

$$d_i = \left(1 + (n-1)\frac{\Delta}{M}\right)\alpha_i + \left(1 - \frac{\Delta}{M}\right)\sum_{j \neq i} \alpha_j + \sum_j (\beta_{ji} - \beta_{ij})$$

Using the fact that $\sum_i \alpha_i = 1$, this becomes

$$d_i = \frac{\Delta}{M}(n\alpha_i - 1) + 1 + \sum_j (\beta_{ji} - \beta_{ij}) \tag{22}$$

When $i \in I$, each $\beta_{ij} = 0$, and we immediately get from (22) that $d_i = a_i$. When $i \in J$, (22) becomes

$$d_i = \frac{n - a(I)}{n - |I|} - \sum_{j \in K} \frac{1}{|K|}\left(\frac{n - a(I)}{n - |I|} - a_i\right) = a_i$$

When $i \in K$, (22) becomes

$$\begin{aligned} d_i &= \frac{n - a(I)}{n - |I|} + \sum_{j \in J} \frac{1}{|K|} \left(\frac{n - a(I)}{n - |I|} - a_j \right) + \sum_{j \in K \setminus \{i\}} (f_{ji} - f_{ij}) \\ &= \left(1 + \frac{|J|}{|K|} \right) \frac{n - a(I)}{n - |I|} - \frac{a(J)}{|K|} + \sum_{j \in K \setminus \{i\}} (f_{ji} - f_{ij}) \end{aligned}$$

Using the fact that $a(J) = n - a(I) - a(K)$, this simplifies to

$$d_i = \frac{a(K)}{|K|} + \sum_{j \in K \setminus \{i\}} (f_{ji} - f_{ij}) \tag{23}$$

But this implies $d_i = a_i$, because the second term is the net supply at vertex i, which is $a_i - a(K)/|K|$.

We conclude that $z \leq au + (n-1)\Delta$ is the linear combination of inequalities in (20) using the multipliers (21). Since $b \geq (n-1)\Delta$, $z \leq au + b$ is dominated by a surrogate of (20) and therefore implied by (9), provided we show that the multipliers in (21) are nonnegative.

We observe first that $\alpha_i \geq 0$ for $i \in I$ because $a_i \geq 1 - \Delta/M$, due to the definition of I. To show that $\alpha_i \geq 0$ for $i \notin I$, we note that $a_i \leq 1$ for $i \in I$ implies $\alpha_i \leq 1/n$, from the definition of α_i. Thus $\alpha(I) \leq 1$, which implies $\alpha_i \geq 0$ for $i \notin I$. To show that $\beta_{ij} \geq 0$ for $i \in J$ and $j \in K$, we note that $a_i \leq 1$ for $i \in I$ implies that $a(I) \leq |I|$, whence

$$\frac{n - a(I)}{n - |I|} \geq 1 \tag{24}$$

But $a_i < 1 - \Delta/M$ for $i \in J \setminus \{j\}$ implies $a_i \leq 1$, which along with (24) implies that $\beta_{ij} \geq 0$. Finally, $\beta_{ij} = f_{ij}$ for $i, j \in K$ is by definition a nonnegative flow.

10 A Numerical Example

We will consider a population made up of five categories $\{1, 2, \ldots, 5\}$. Policies are to be pursued that will result in each category i having utility u_i. These policies will be represented by the following constraints on the utilities.

$$\begin{aligned} &4u_2 + 5u_3 + u_5 \leq 30 \\ &6u_1 + 2u_3 + u_4 \leq 20 \\ &u_1 + u_2 + u_3 \geq 5 \end{aligned}$$

The first two constraints can be regarded as arising from resource limitations and the third constraint from a policy stipulation. Note that the available resources have the greatest utility benefits when allocated to categories 4 and 5. These might regarded as patient categories whose ailments are more easily cured or ameliorated. As in the model above, we introduce a "big M" condition $|u_1 - u_2| \leq 100$.

If we were to pursue a utilitarian policy of maximising the sum of utilities, the following distribution of utililities would result:

$$(u_1, \ldots, u_5) = (0, 5, 0, 20, 10)$$

The total (maximised) utility is 35, but there is an unacceptable level of "inequity," particularly in that two categories are completely left out of the resource allocation.

In contrast, if we were to pursue a purely Rawlsian policy of maximising the minimum utility, the following distribution of utilities would result:

$$(u_1, \ldots, u_5) = (2.22, 2.22, 2.22, 2.22, 10)$$

Much of the disparity has been removed, but at the expense of reducing total utility to 18.89.

We now solve the model (9) for various values of Δ, so that only if disparity rises above Δ will a (partially) utilitarian objective be invoked. We find that for $\Delta \geq 5.56$, the solution switches from the utilitarian one to

$$(u_1, \ldots, u_5) = (2.78, 1.11, 1.11, 1.11, 20)$$

This differs from the utilitarian solution in that no one is left out entirely, and yet the resulting total utility of 26.11 is substantially higher than in a purely Rawlsian solution. Note that utility is allocated among the categories in a way that would have been hard to predict. When $\Delta \geq 13.34$, the solution becomes purely Rawlsian.

11 Variants of the Problem

Variants of our model are clearly possible. For example:

(i) Instead of working with a fixed Δ, we could vary this with the magnitude of the values of the u_j.

(ii) The objectives $\min_j\{u_j\}$ and $\sum_j u_j$ could be added in suitable multiples to give a composite objective. This would avoid the need for a MIP model but obfuscate the central dilemma of equity versus utilitarianism.

Many other variants are possible, but our main aim is to establish the model presented here first.

12 Other relevant papers

There have been a number of papers suggesting mathematical programming models for the allocation of health resources. We will not list them all here. One such is Stinnett and Paltiel (1995).

Optimization models can be formulated for purely utilitarian and purely Rawlsian distributions. Structural properties of the solutions are derived in Hooker (to appear), which interprets the Rawlsian criterion as a lexicographic maximum.

The problem of MIP representability was first analysed by Jeroslow (1987, 1989). His results are extended in Williams (2009) and Hooker (2009).

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