Graph Coloring Facets from All-different Systems

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Motivation

- **Integer programming** often encodes discrete choices with **0-1 variables.**
  - Convenient for obtaining a **continuous relaxation** and strong **cutting planes**.

Motivation

- **Constraint programming** formulations often use **finite-domain variables**.
  - x_i = finite domain variable
    - Job assigned to worker i
    - Start time of job i
    - City visited after city i
  - y_{ij} = corresponding 0-1 variable
    - $y_{ij} = 1$ if $x_i = j$

Motivation

- A finite-domain formulation can provide a (smaller) continuous relaxation.
  - If the domains are **numeric**.
  - The **polyhedral structure** is very different from the 0-1 model.
  - **Finite-domain cuts** can be mapped into the 0-1 model.
  - This may yield **stronger cuts** in the 0-1 model.

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- A finite-domain formulation can provide a (smaller) continuous relaxation.
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  - The **polyhedral structure** is very different from the 0-1 model.
  - **Finite-domain cuts** can be mapped into the 0-1 model.
  - This may yield **stronger cuts** in the 0-1 model.
- We apply this idea to **graph coloring**.
  - Has a natural CP formulation.

Motivation

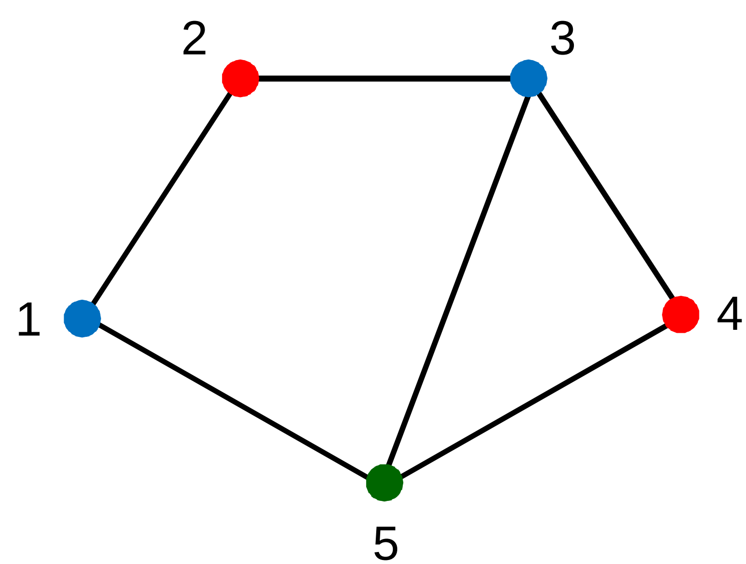
- We obtain two kinds of results:
  - If you find a structure (e.g., odd hole) that yields a known valid inequality in 0-1 space…
    - We will give you a stronger cut for **free**.
    - Use whatever separation algorithm you want.

Motivation

- We obtain two kinds of results:
  - If you find a structure (e.g., odd hole) that yields a known valid inequality in 0-1 space…
    - We will give you a stronger cut for **free**.
    - Use whatever separation algorithm you want.
  - We identify **additional** structures that yield valid inequalities.
    - They are **stronger** than **known cuts**.
    - Many **fewer** are required.
    - We have separation algorithms (if needed).

Graph Coloring

- We focus on the **vertex coloring** problem.
  - Given a graph, assign colors to vertices so that no two adjacent vertices receive the same color.
  - Minimize the number of colors.



Graph Coloring

- 0-1 model

$$\min \sum_j w_j$$

= 1 if color j is used

$$\sum_j y_{ij} = 1, \quad \text{all vertices } i$$

$$y_{1j} + y_{2j} \le w_j, \quad \text{all colors } j$$

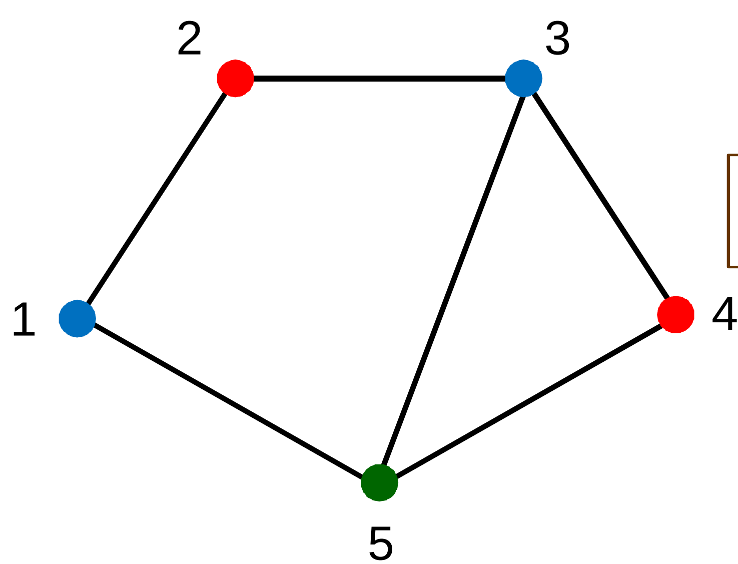
$$y_{1j} + y_{5j} \le w_j, \quad \text{all colors } j$$

$$y_{2j} + y_{3j} \le w_j, \quad \text{all colors } j$$

$$y_{3j} + y_{4j} + y_{5j} \le w_j, \quad \text{all colors } j$$

$$y_{ij} \in \{0,1\}$$

= 1 if vertex i receives color j



Graph Coloring

- General model:

= 1 if color j is used

$$\min \sum_j w_j$$

$$\sum_j y_{ij} = 1, \quad \text{all vertices } i$$

$$\sum_{i \in V_k} y_{ij} \le w_j, \quad \text{all colors } j \text{, cliques } V_k \text{ that cover vertices}$$

$$y_{ij} \in \{0,1\}$$

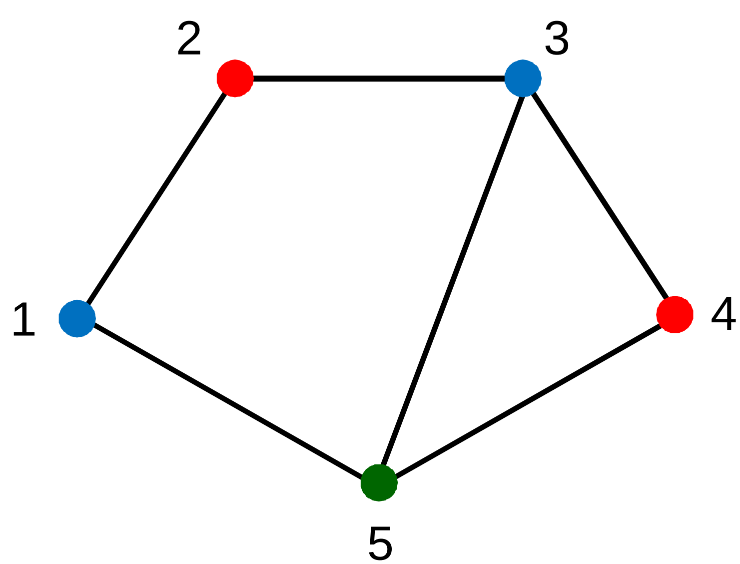
= 1 if vertex i receives color j

$O(n^2)$ variables

$O(n^3)$ constraints

Alldiff Systems

- Use an **all-different** constraint for each clique.



$$
\begin{aligned}
&\min\ z \\
&z \ge x_i, \quad \text{all vertices } i \\
&\text{alldiff}(x_1, x_2), \quad \text{all colors } j \\
&\text{alldiff}(x_1, x_5), \quad \text{all colors } j \\
&\text{alldiff}(x_2, x_3), \quad \text{all colors } j \\
&\text{alldiff}(x_3, x_4, x_5), \quad \text{all colors } j \\
&x_i \in \{1, \ldots, 5\}
\end{aligned}
$$

x_i = color assigned to vertex i

Alldiff Systems

- General model:

$$\begin{aligned}&\min z\\&z \geq x_i, \quad \text{all vertices } i\\&\text{alldiff}\left(x_i \mid i \in V_k\right), \text{ all cliques } V_k\\&\boxed{x_i} \in \{1,\ldots,n\}\end{aligned}$$

= color assigned to vertex *i*

O(*n*) variables
O(n^2) constraints

Objective reduces **symmetry**

Alldiff Systems

- Applications:
  - Scheduling, timetabling.
    - Employee scheduling.
    - Course timetabling.
  - Latin squares.
    - Alldiff for each row, column.
    - Experimental design: orthogonal Latin squares.
    - Sudoku puzzles.
  - Graph coloring.
    - Many applications.

Related Work

- Convex hull of single alldiff.
  - Hooker (2000), Williams and Yan (2001).
- Convex hull of 2 alldiffs.
  - Appa, Magos and Mourtos (2004)
- Convex hull of alldiff systems with inclusion property.
  - Appa, Magos and Mourtos (2011).
  - Same facets as individual alldiffs.
- Some facets of systems without inclusion property.
  - Magos and Mourtos (2011).

Variable Mapping

- There is a linear mapping from x_i to y_{ij}:

$$x_i = \sum_j j y_{ij}$$

- Any valid linear inequality in x_i-space maps to a valid linear inequality in y_{ij}-space.
  - Just substitute above expression for x_i.
  - Convert any finite domain cut to a 0-1 cut.

Choice of Domain

- We will assume each x_i has domain $\{0, \ldots, n - 1\}$.
  - To simplify exposition.

- Most results to follow can be generalized to an arbitrary numeric domain $\{v_1, \ldots, v_n\}$ with each $v_i \geq 0$.
  - Some results are valid for domain $D = \{0, \delta, \ldots, (n - 1)\delta\}$ with $\delta > 0$.

Single Alldiff

- The polytope defined by the single alldiff constraint

$$\text{alldiff}(x_1, x_2, x_3) \qquad x_i \in \{0,1,2\}$$

has facets $\quad x_1, x_2, x_3 \geq 0$

$$x_1 + x_2 \geq 1, \quad x_1 + x_3 \geq 1, \quad x_2 + x_3 \geq 1,$$

$$x_1 + x_2 + x_3 = 3$$

Single Alldiff

- The facet-defining inequality

$$x_1 + x_2 \geq 1$$

  maps to the 0-1 inequality

$$y_{11} + 2y_{12} + y_{21} + 2y_{22} \geq 1$$

- This is not facet-defining because the convex hull of the feasible set has dimension 4, while only 2 points lie on the face:

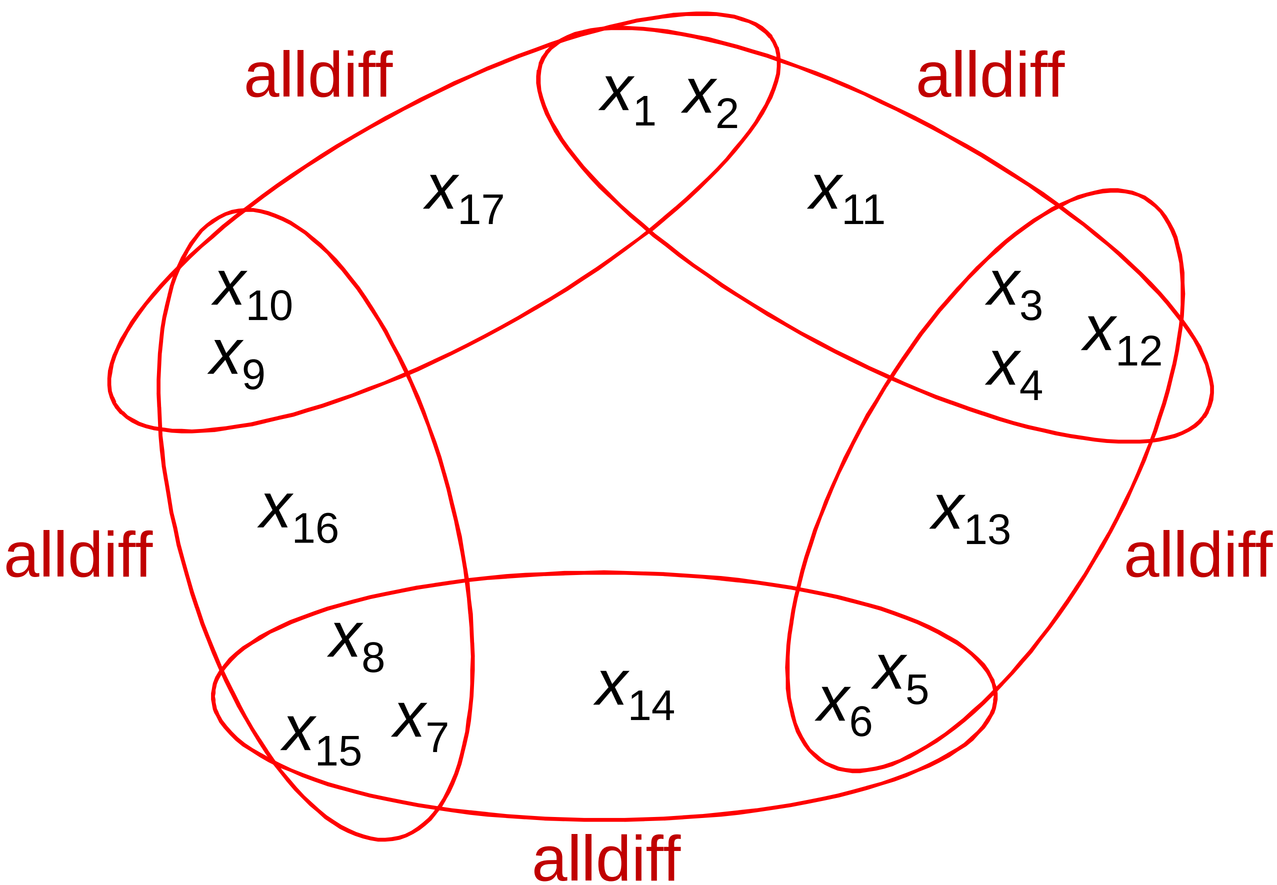
$$\begin{bmatrix} y_{10} & y_{11} & y_{12} \\ y_{20} & y_{21} & y_{22} \\ y_{30} & y_{31} & y_{32} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Single Alldiff

- In general, facet-defining finite domain cuts don’t map to facet-defining 0-1 cuts.
- The 0-1 cuts can nonetheless be significantly stronger than known cuts.

Odd Cycles

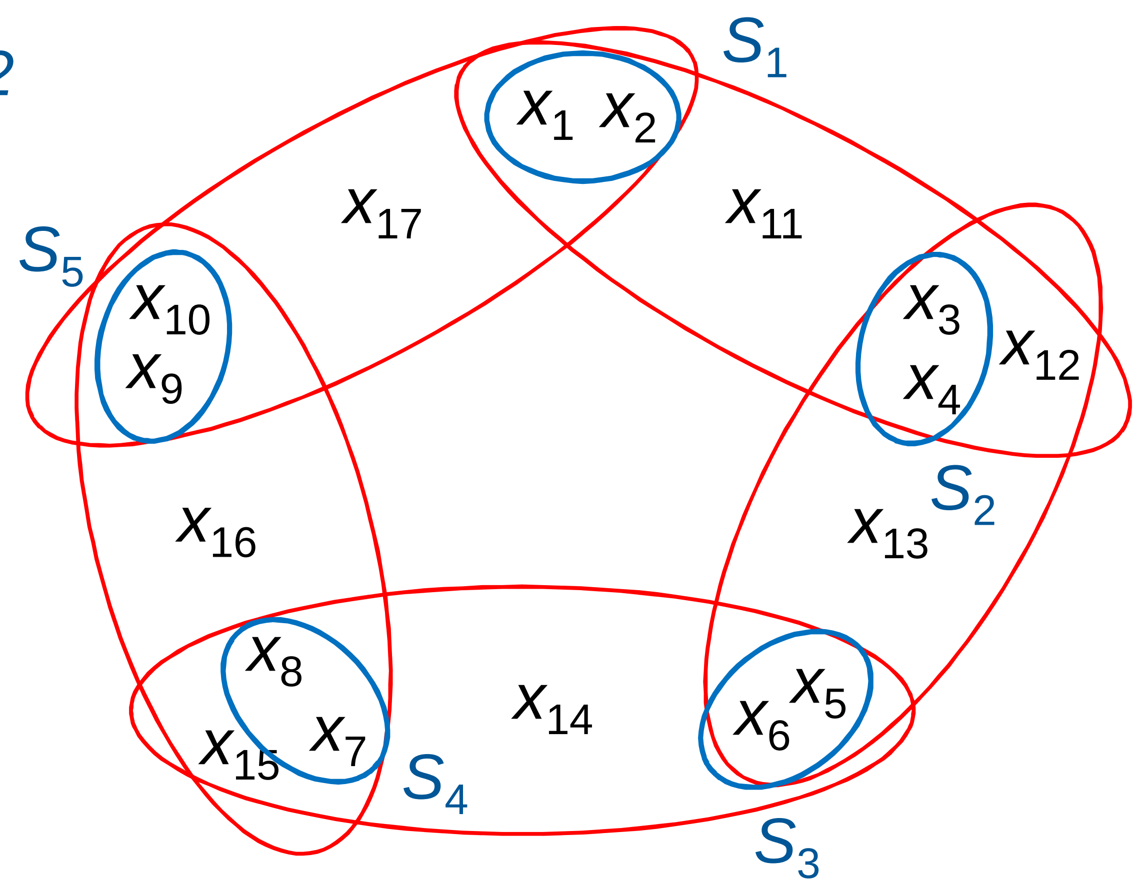
- A *q*-cycle consists of *q* alldiff constraints that look like this:



Odd Cycles

- Select any subset of s vertices in each overlap:

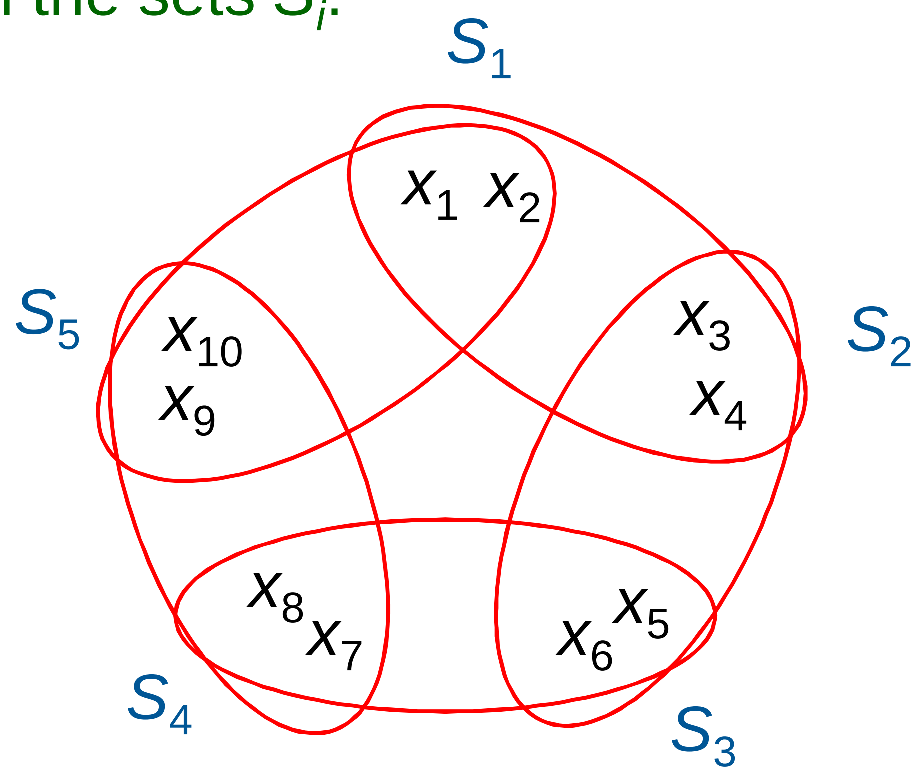
$s = 2$



Odd Cycles

- Focus on the sets S_i:

$s = 2$
$q = 5$



$sq = 10$ vertices
Each color can be assigned to at most $(q - 1)/2 = 2$ vertices.

We need at least $L = \left\lceil \frac{sq}{(q-1)/2} \right\rceil = 5$ colors

Odd Cycles

- Focus on the sets S_i:

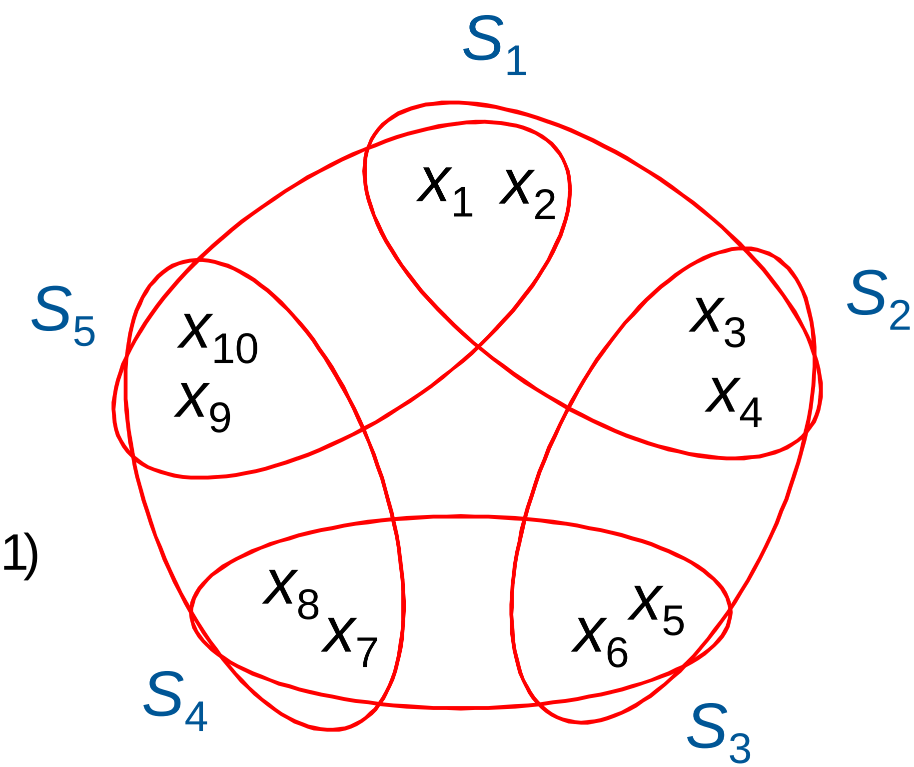
$s = 2$

$q = 5$ $\quad S = \bigcup_k S_k$

So

$$\sum_{i \in S} x_i \geq \frac{q-1}{2} \cdot 0 + \frac{q-1}{2} \cdot 1 + \cdots$$

$$\cdots + \frac{q-1}{2}(L-2) + \left(sq - \frac{q-1}{2}(L-1) \right)(L-1)$$



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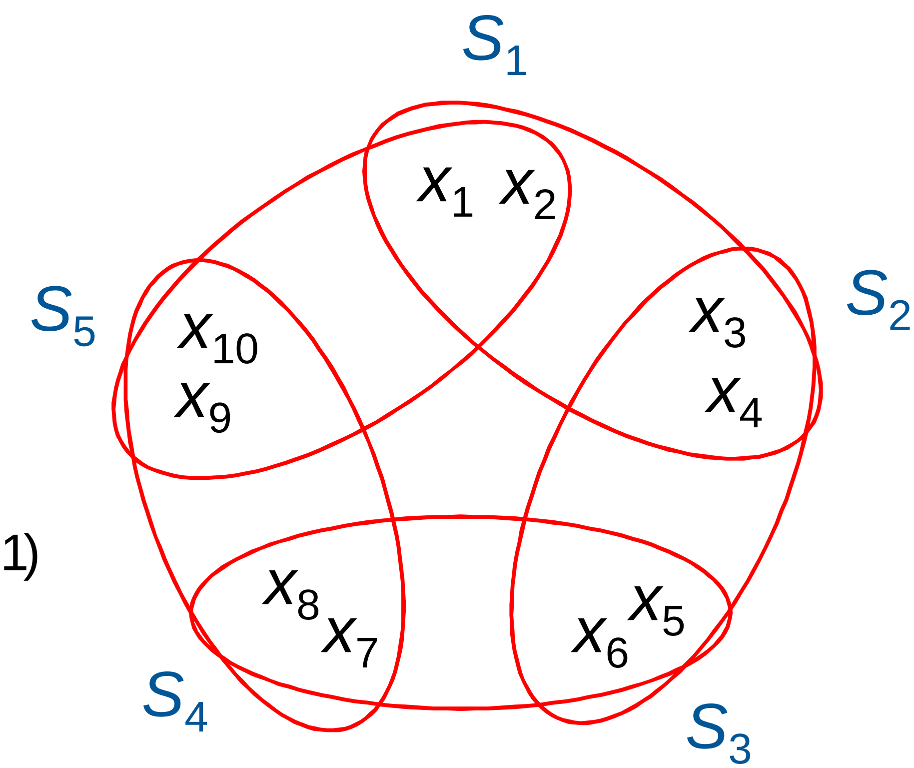
$$= \left(sq - \frac{q-1}{4} L \right)(L-1) = 20$$



Odd Cycles

- So we have a valid inequality:

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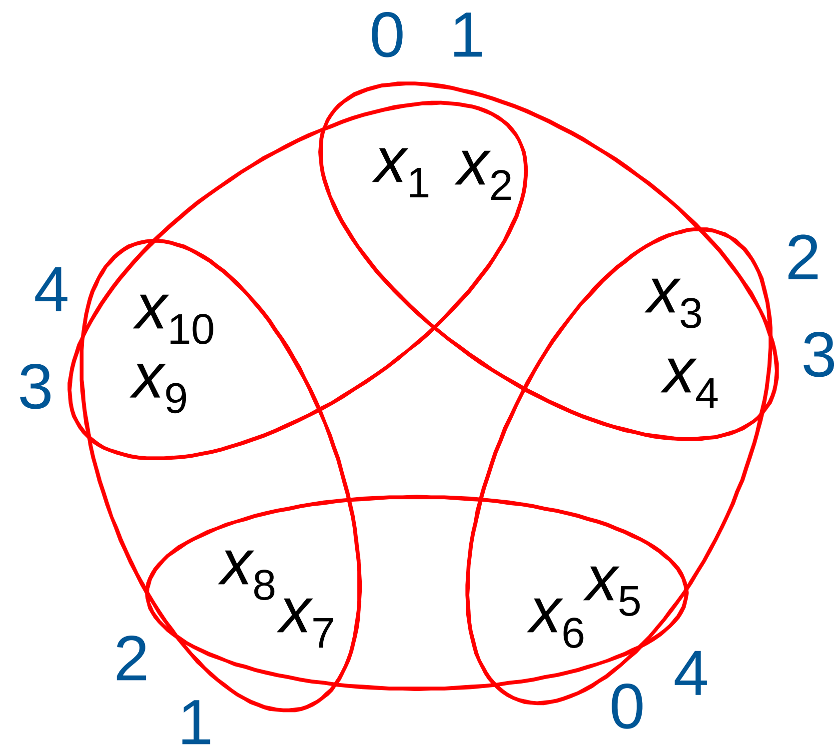
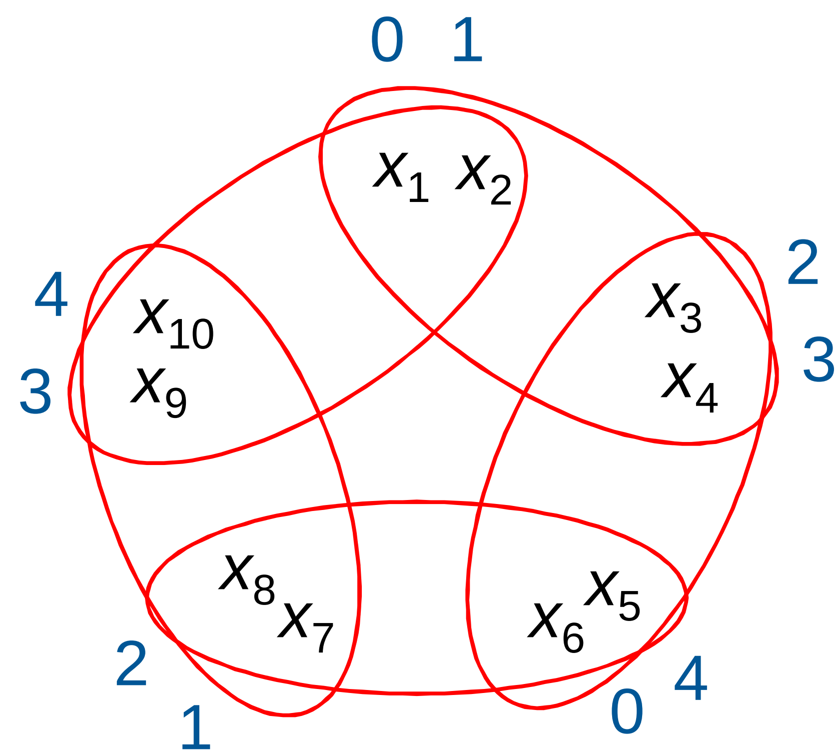


Odd Cycles

- So we have a valid inequality:

$$\sum_{i \in S} x_i \geq \left(sq - \frac{q-1}{4} L \right)(L-1) = 20$$

- The inequality is **facet-defining** if *q* is odd.
  – and if the *q*-cycle is the subgraph induced by vertices in the cycle.



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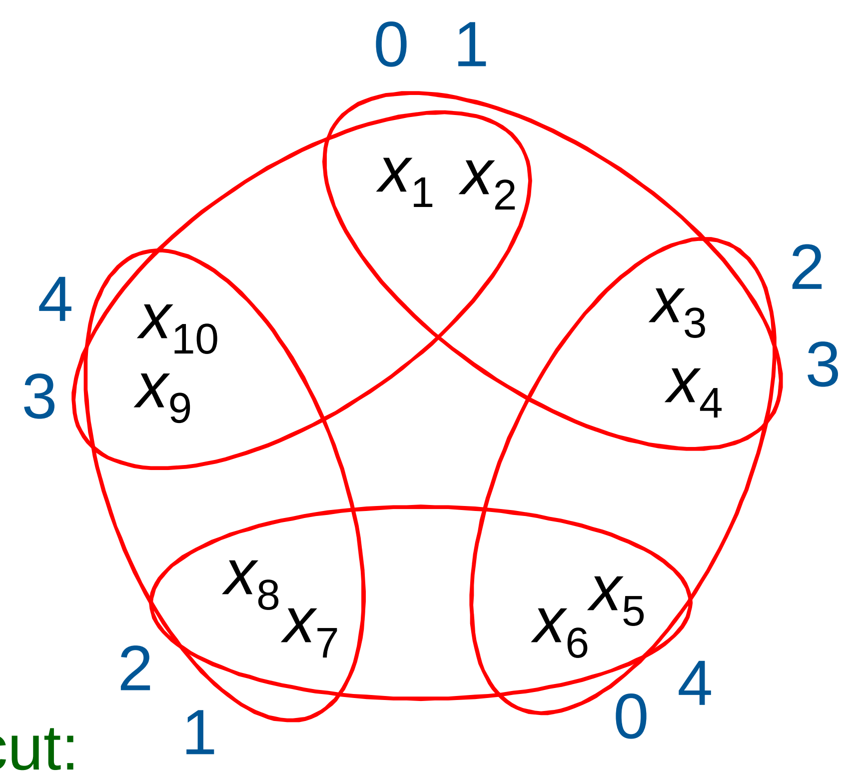


- The inequality is **facet-defining** if q is odd.
  - and if the q-cycle is the subgraph induced by vertices in the cycle.
- For $s = 1$, we have odd hole cut:

$$\sum_{i \in S} x_i \geq \frac{q+3}{2}$$

Odd Cycles

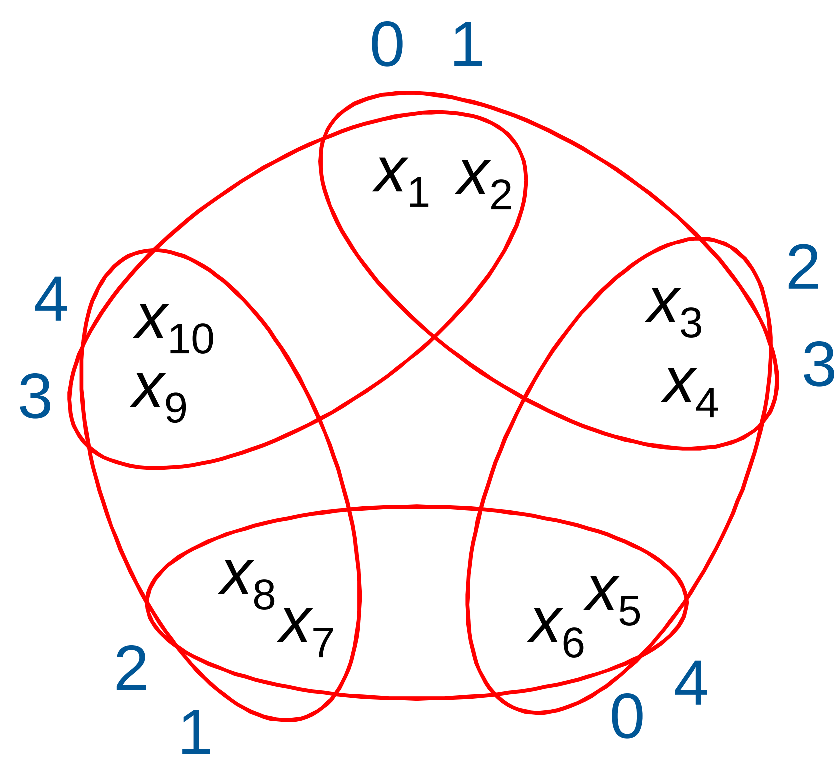
- So we have a valid inequality:

$$\sum_{i \in S} x_i \geq \left(sq - \frac{q-1}{4} L \right)(L-1) = 20$$

- We can obtain a valid bound on number of colors z by substituting $z - x_i$ for x_i:

$$z \geq \frac{1}{qs} \sum_{i \in S} x_i + \left(1 - \frac{q-1}{4qs} L \right)(L-1)$$

$$= \frac{1}{10} \sum_{i \in S} x_i + 2$$



This is **facet defining** for domain *D*.

z-cuts in general

- In fact, facet-defining **x**-cuts for a graph coloring problem always give rise to facet-defining **z**-cuts:
  - **Theorem:** if $ax \geq b$ is facet defining for a coloring problem with domain $D = \{0, \delta, 2\delta, \ldots, (n-1)\delta\}$ for $\delta > 0$, then $aez \geq ax + b$ is also facet defining, where $e = (1, \ldots, 1)$.

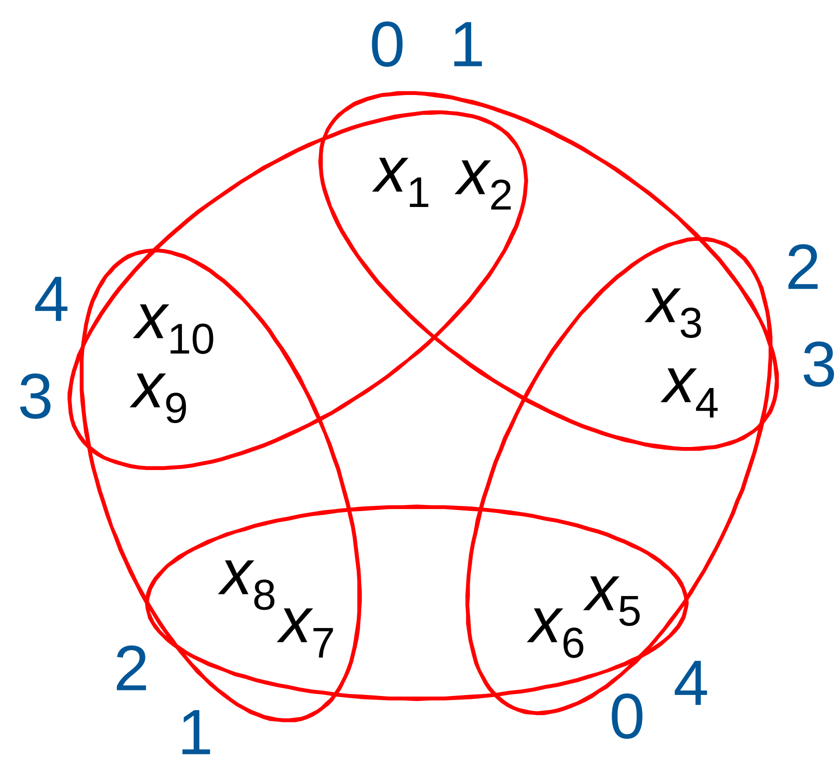
Mapping into 0-1 Space

- The **x-cut**

$$\sum_{i \in S} x_i \geq \left(sq - \frac{q-1}{4} L \right)(L-1) = 20$$

maps into a 0-1 cut
by replacing x_i with $\sum_j j y_{ij}$

  - How does it compare
    with classical odd hole cuts?



Mapping into 0-1 Space

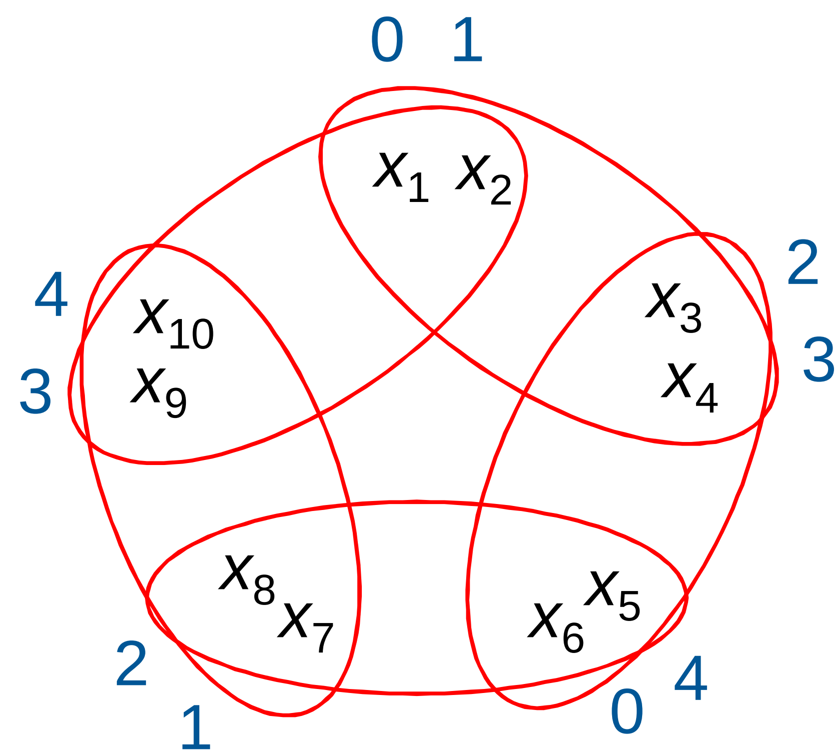
- The **z-cut**

$$z \geq \frac{1}{qs}\sum_{i \in S} x_i + \left(1 - \frac{q-1}{4qs}L\right)(L-1)$$

maps into a 0-1 cut
by replacing x_i with $\sum_j j y_{ij}$

and z with $\sum_j w_j - 1$

  – How does it compare
  with classical odd hole cuts?

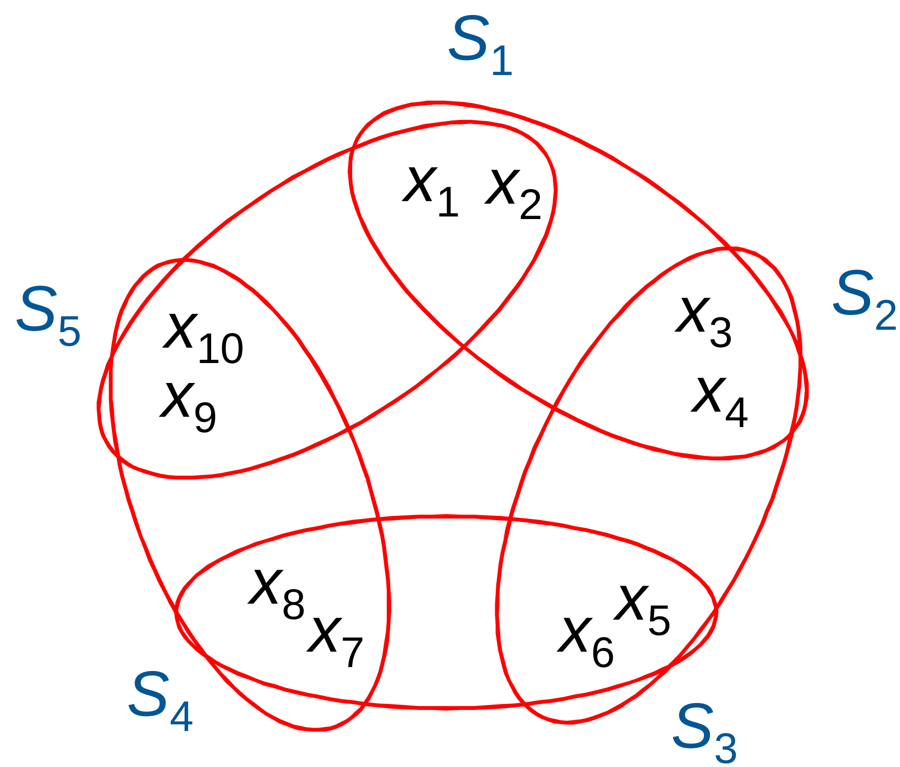


Comparison with Odd Hole Cuts

- A q-cycle defines $s^q = 32$ odd hole cuts for each color:

$$\sum_{i \in T} y_{ij} \le \frac{q-1}{2} w_j, \text{ all } T, j$$

  - where T selects one vertex from each S_k



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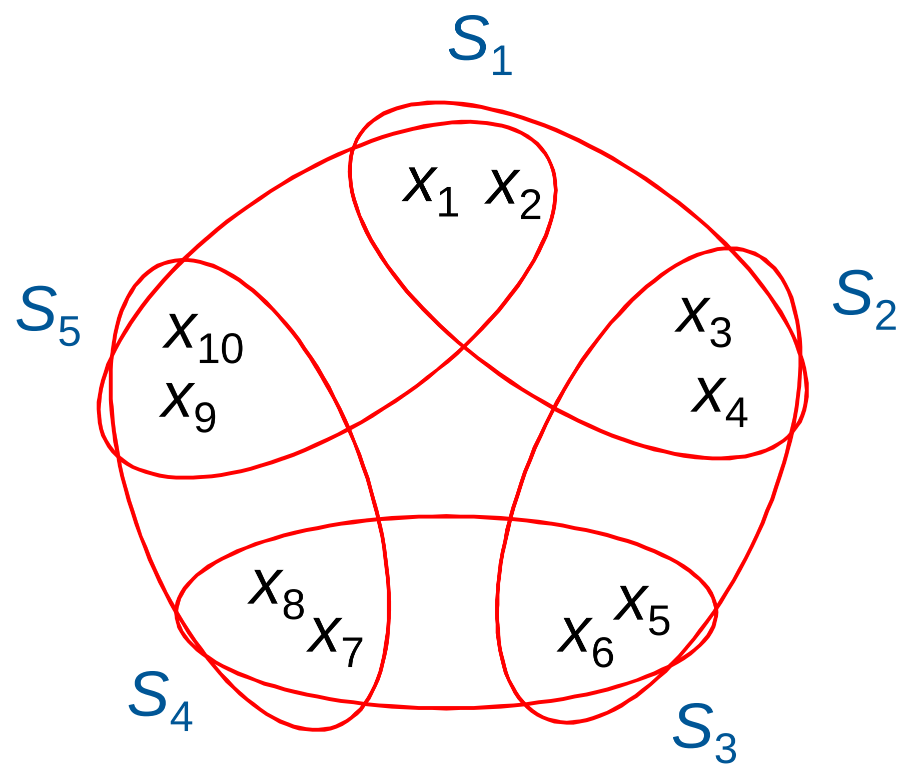
- For $s \ge 2$, one **x**-cut is stronger than all of these odd hole cuts.



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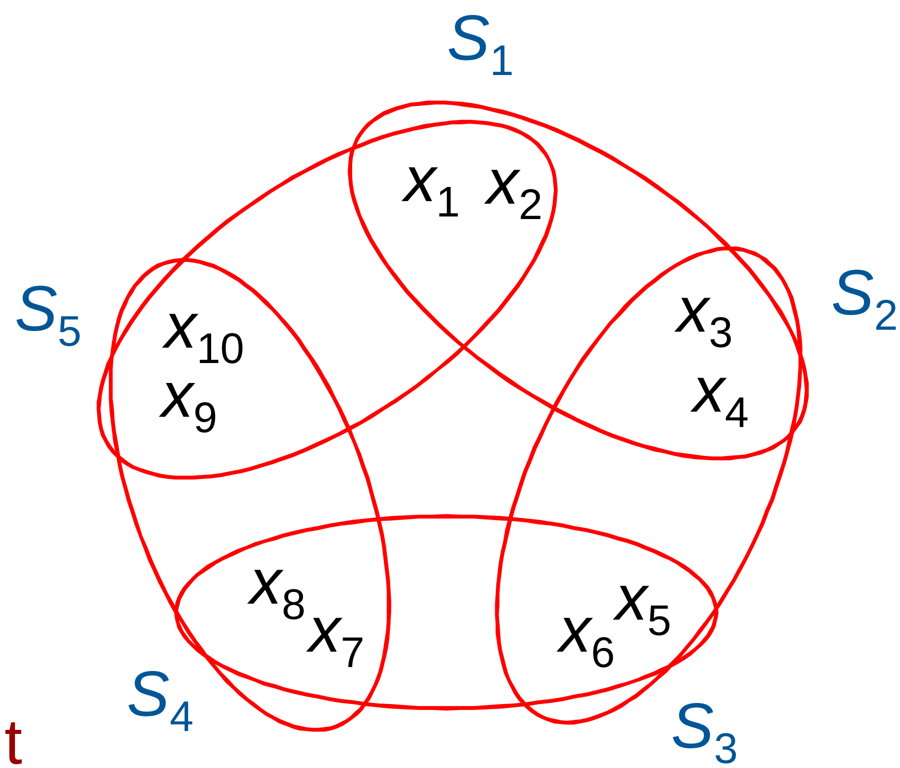
  - where T selects one vertex from each S_k
- For $s \ge 2$, one **x**-cut is stronger than all of these odd hole cuts.
  - Adding a **z**-cut to the **x**-cut tightens the bound further.



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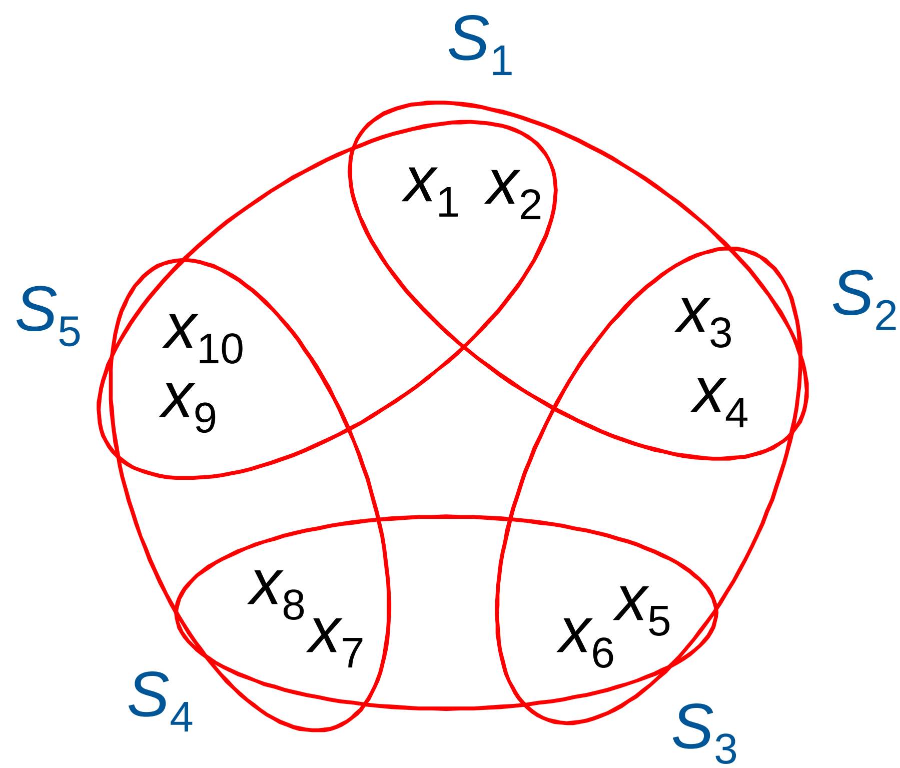
- For **any** s (including $s = 1$), one **x**-cut and one **z**-cut are stronger than all of these odd hole cuts.



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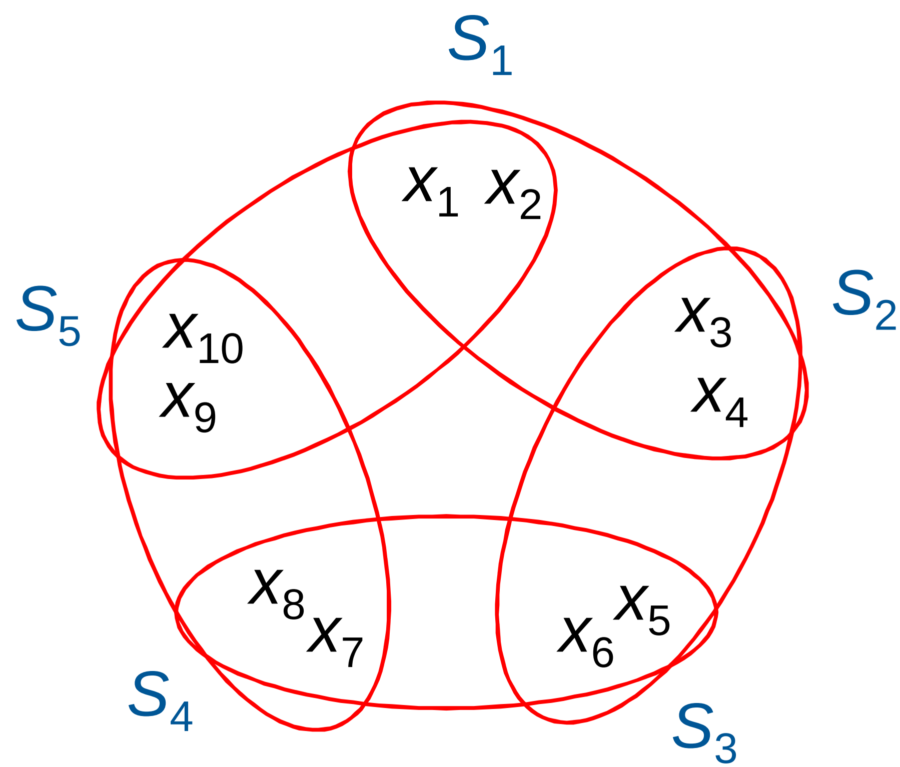
  – where T selects one vertex from each S_k

- For **any** s (including $s = 1$), one **x**-cut and one **z**-cut are stronger than all of these odd hole cuts.

  – For any separating odd hole cut, replace it with **x**-cut and **z**-cut for $s = 1$ to get a stronger cut.



Computed Bounds

Lower bound on number of colors in
0-1 model of 5-cycle

s **=**	**1**	**2**	**3**	**4**	**5**
All odd hole cuts*	2.5	4.0	6.0	8.0	10.0
x-cut only	2.0	4.0	6.0	8.0	10.0
z-cut only	2.3	4.5	6.77	9.0	10.0
x and **z**-cut only	2.6	5	7.53	10	12.52
Optimal	3	5	8	10	13
No. odd hole cuts	5	320	3645	20,480	78,125

* And clique inequalities

Computed Bounds

Lower bound on number of colors in
0-1 model of 7-cycle

s =	1	2	3	4
All odd hole cuts*	2.33	4.0	6.0	8.0
x-cut only	2.0	4.0	6.0	8.0
z-cut only	2.21	4.36	6.5	8.68
x and **z**-cut only	2.43	4.71	7	9.36
Optimal	3	5	7	10
No. odd hole cuts	7	1792	45,927	458,752

* And clique inequalities

Computed Bounds

Lower bound on number of colors in
0-1 model of 9-cycle

***s* =**	**1**	**2**	**3**
All odd hole cuts*	2.25	4.0	6.0
x-cut only	2.0	4.0	6.0
z-cut only	2.17	4.28	6.39
x and **z**-cut only	2.33	4.56	6.78
Optimal	3	5	7
No. odd hole cuts	9	9612	531,441

* And clique inequalities

Cuts in x-space

- Finite domain cuts can also be used in their original form.
  - This results in a much more compact relaxation.
    - $O(n)$ variables rather than $O(n^2)$ variables.
- Is the bound in the x-space as tight as in the 0-1 space?

Cuts in x-space

- Finite domain cuts can also be used in their original form.
  - This results in a much more compact relaxation.
    - $O(n)$ variables rather than $O(n^2)$ variables.
- Is the bound in the x-space as tight as in the 0-1 space?
  - Yes.

Computed Bounds

Lower bound on number of colors in
x-model of 5-cycle

s =	**1**	**2**	**3**	**4**	**5**
Clique cuts only	1.5	2.5	3.5	4.5	5.5
Plus **x**-cut	1.8	3.0	4.27	5.5	6.76
Plus **z**-cut	2.3	4.5	6.77	9.0	11.26
Plus **x** and **z**-cut	2.6	5	7.53	10	12.52
Optimal	3	5	8	10	13

Computed Bounds

Lower bound on number of colors in
x-model of 7-cycle

***s* =**	**1**	**2**	**3**	**4**
Clique cuts only	1.5	2.5	3.5	4.5
Plus **x**-cut	1.71	2.86	4.0	5.18
Plus **z**-cut	2.21	4.36	6.5	8.68
Plus **x** and **z**-cut	2.43	4.71	7	9.36
Optimal	3	5	7	10

Computed Bounds

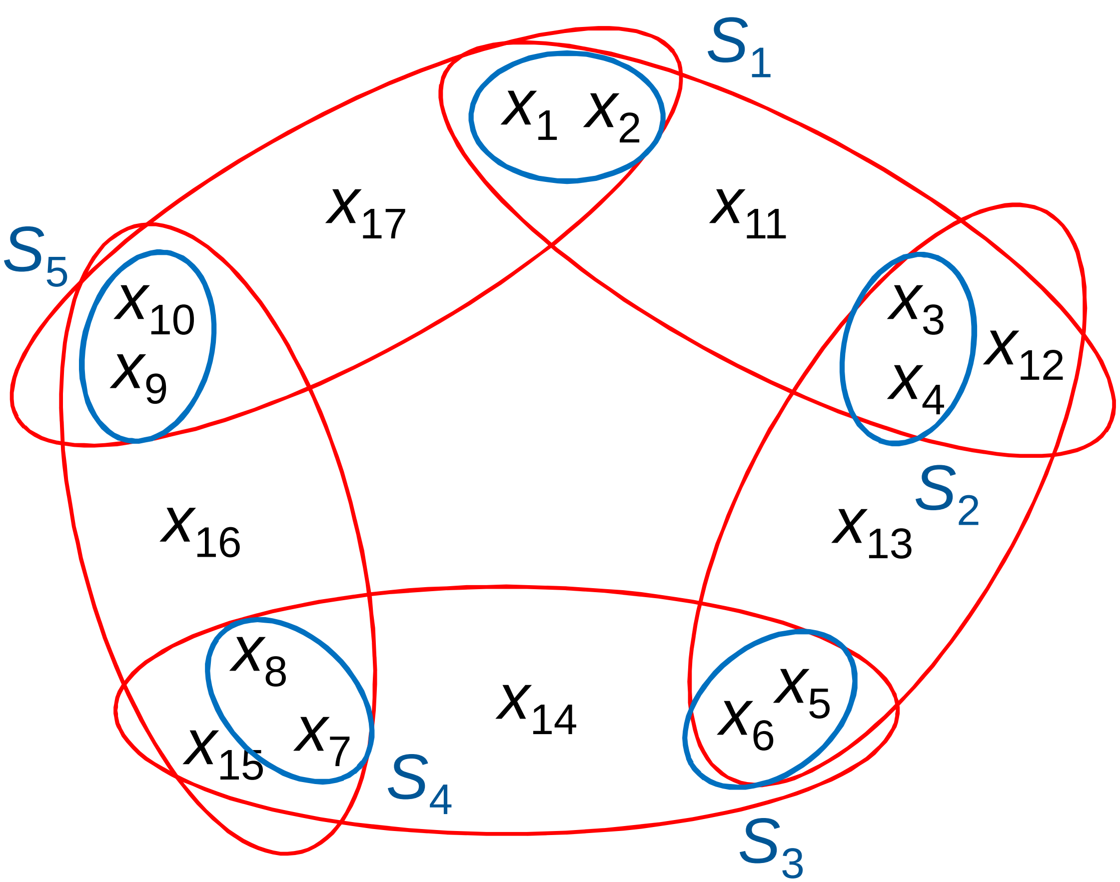
Lower bound on number of colors in *x*-model of 9-cycle

s **=**	**1**	**2**	**3**
Clique cuts only	1.5	2.5	3.5
Plus **x**-cut	1.67	2.78	3.89
Plus **z**-cut	2.17	4.28	6.39
Plus **x** and **z**-cut	2.33	4.56	6.78
Optimal	3	5	7

Separation Heuristic

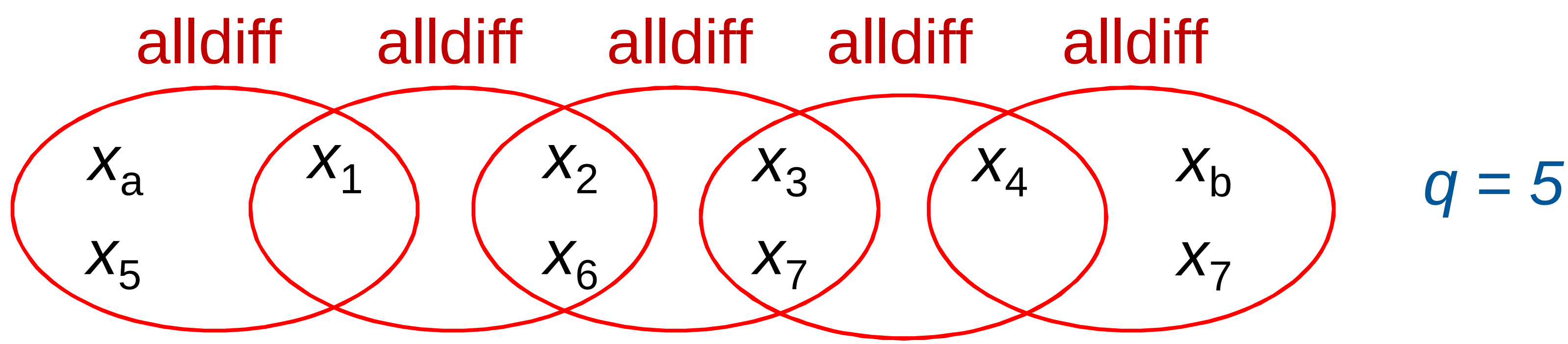
- Select subset of s vertices in each overlap with smallest values in current relaxation:

$s = 2$



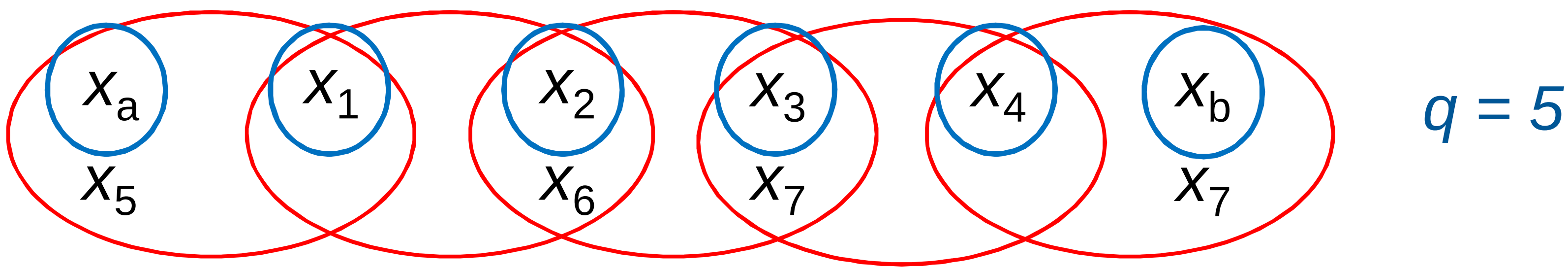
Odd Paths

- A q-path looks like



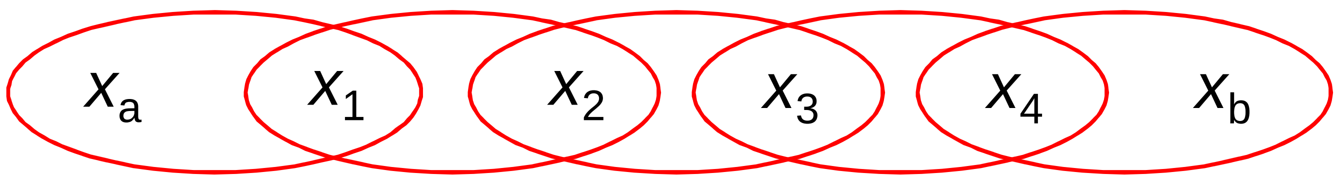
Odd Paths

- Select $q + 1$ variables:



Odd Paths

- This yields a valid inequality (**x**-cut)



$q = 5$

0	1	0	1	0	1
0	1	0	1	2	0

$$2(x_a + x_b) + \sum_{i=1}^{q-1} x_i \geq \frac{q+3}{2} = 4$$

Odd Paths

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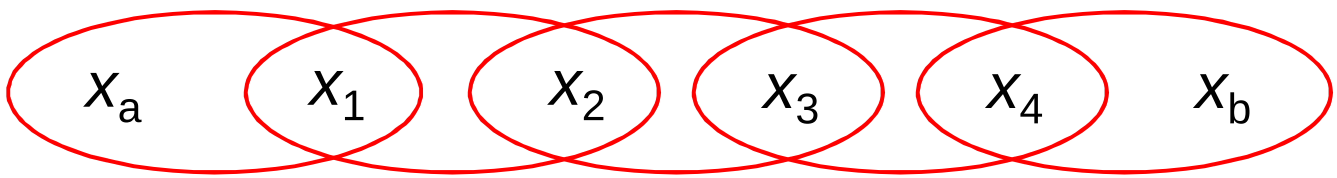

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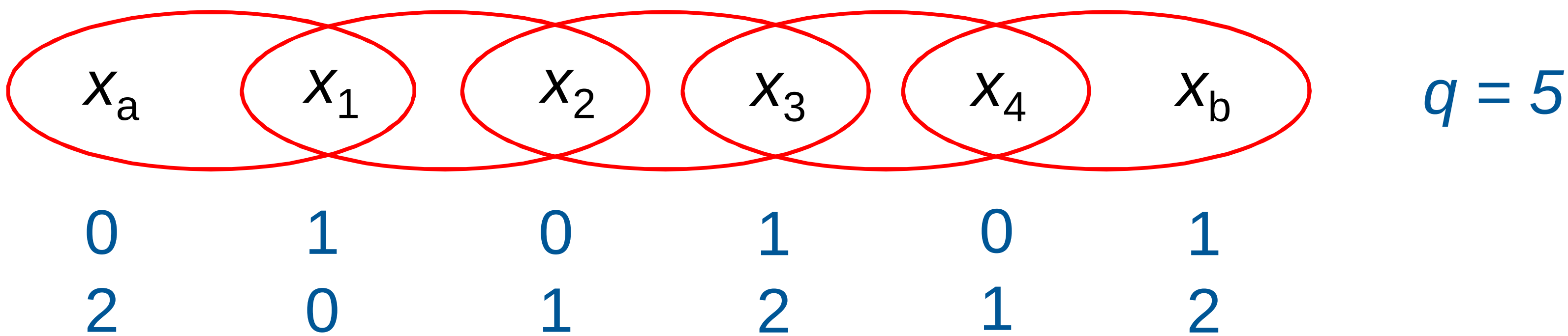
- The inequality is **facet-defining** if q is odd.
  – and if the q-path is the subgraph induced by vertices in the cycle.

Odd Paths

- We also have a **z**-cut



$$z \geq \frac{1}{q+3}\left(2(x_a + x_b) + \sum_{i=1}^{q-1} x_i\right) + \frac{1}{2}$$

- This is also facet defining.

Mapping into 0-1 Space

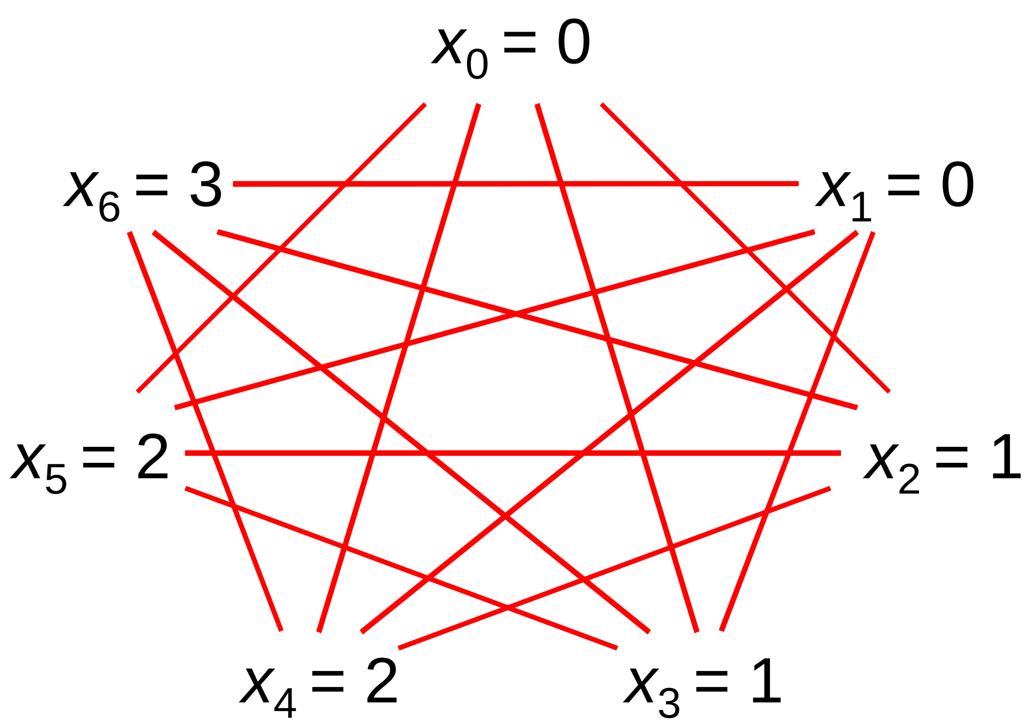
- When mapped into 0-1 space, the finite domain cuts are **redundant** of the 0-1 model.
  - They don't change the bound.

Mapping into 0-1 Space

- When mapped into 0-1 space, the finite domain cuts are **redundant** of the 0-1 model.
  - They don't change the bound.
- However, the finite domain cuts provide a compact relaxation.
  - For each *q*-path, **replace** *q* clique constraints with one **x**-cut and one **y**-cut.
  - Gives the same bound in a problem consisting of one path.

Webs

- A web $W(q,k)$ is a cycle of q vertices in which edges connect all vertices separated by distance at least k.
  - $W(q,2)$ is an anti-hole.



$W(7,2)$

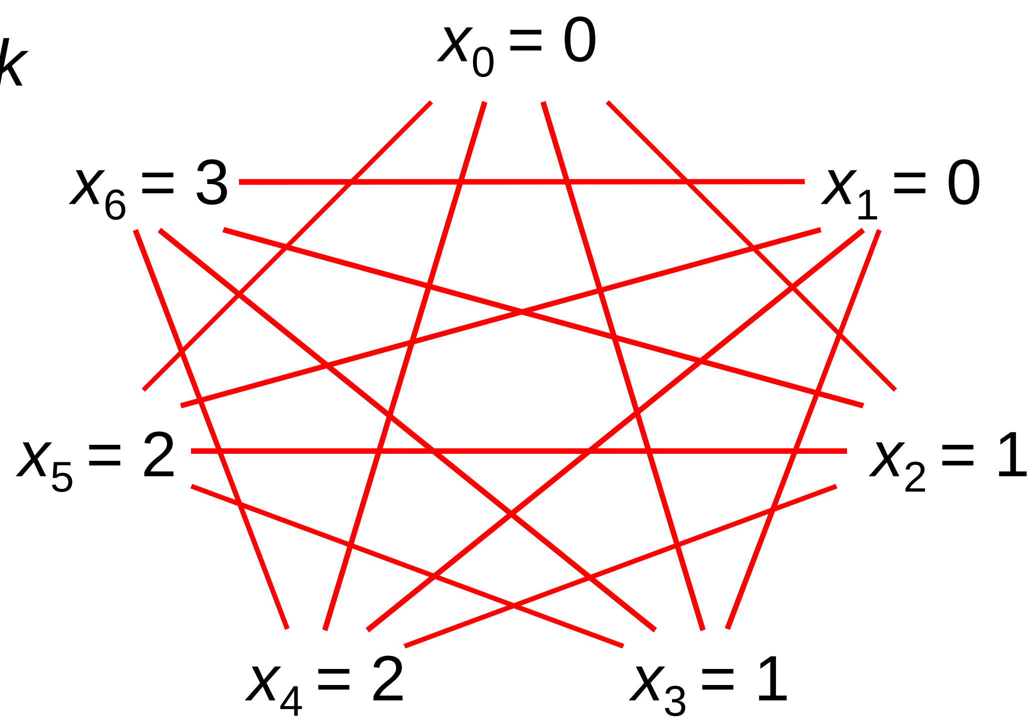
Webs

- If q and k are mutually prime,

$$\sum_i x_i \geq rq - \frac{1}{2}(r+1)rk$$

where $r = \left\lfloor \frac{q}{k} \right\rfloor$

is facet-defining.



$$\sum_i x_i \geq 9$$

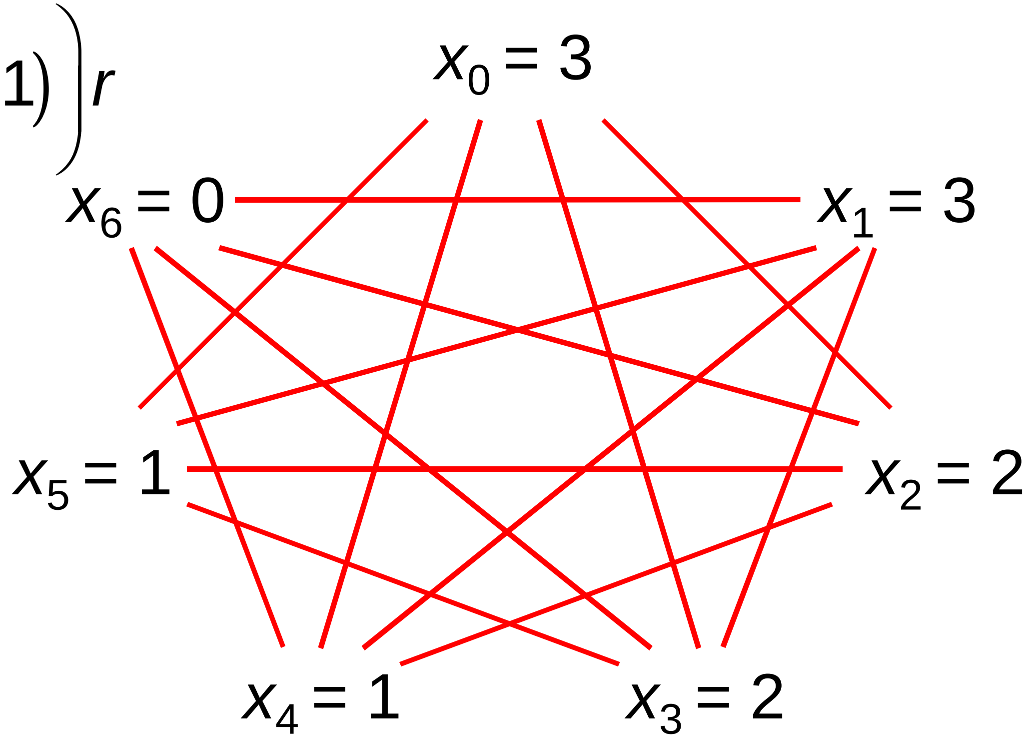
Webs

- If q and k are mutually prime,

$$z \geq \frac{1}{q}\sum_i x_i + \left(1 - \frac{k}{2q}(r+1)\right)r$$

where $r = \left\lfloor \frac{q}{k} \right\rfloor$

is facet-defining.



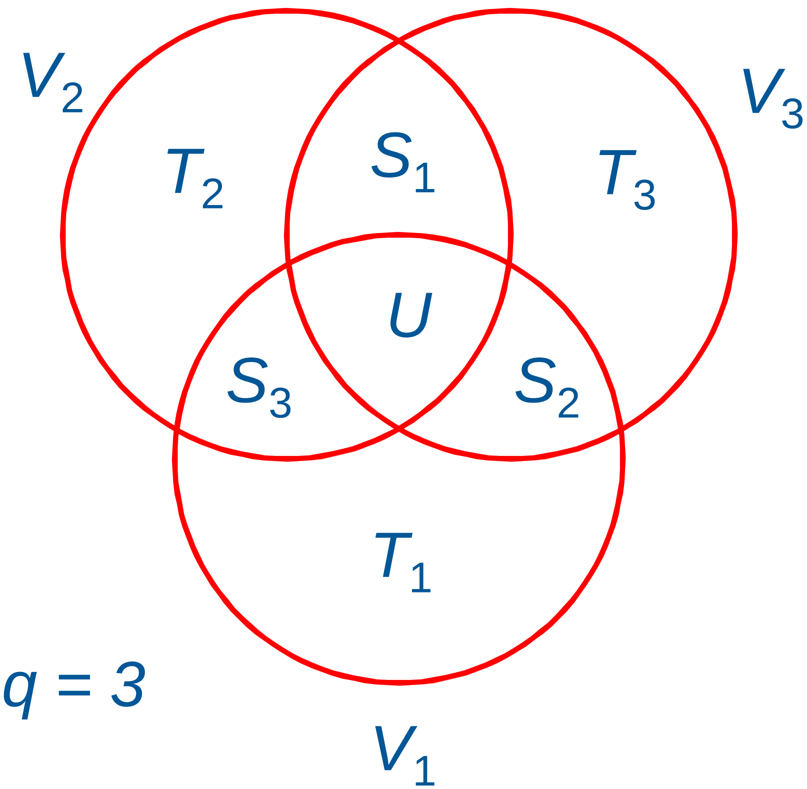
$$\sum_i x_i \geq 9$$

Mapping into 0-1 Space

- Finite-domain **web cuts** perform similarly to finite-domain **odd cycle cuts with $s = 1$**.

Intersecting Systems

- A q-intersecting system looks something like



$q = 3$

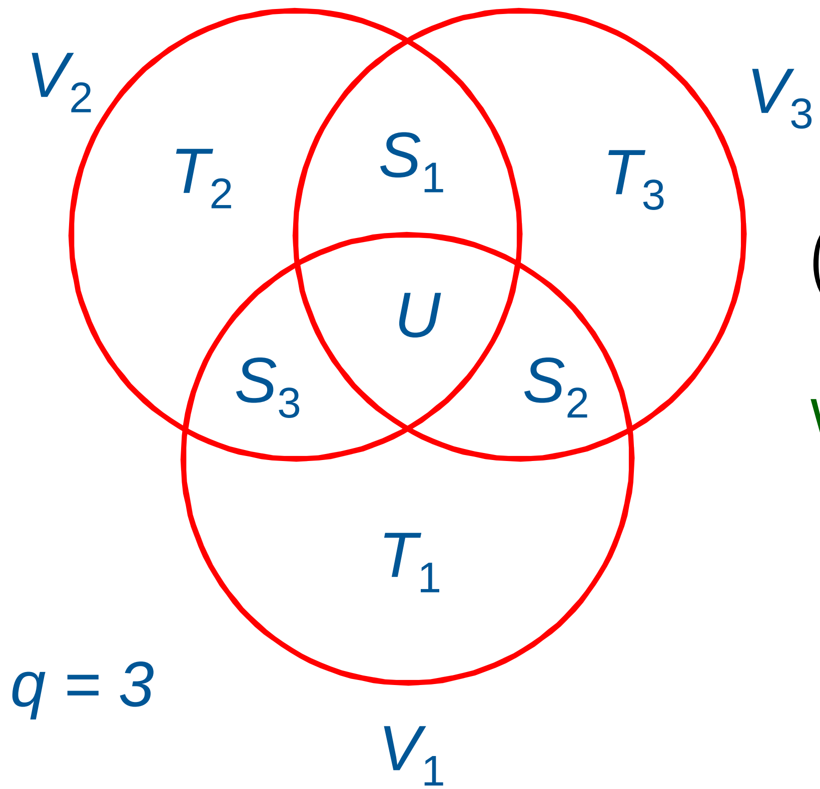
$$S_k = \bigcap_{\ell \neq k} V_\ell \setminus V_k$$

$$T_k = V_k \setminus \bigcup_{\ell \neq k} V_\ell$$

$$U = \bigcap_k V_k$$

Intersecting Systems

- Facet-defining inequality. Let $S = \bigcup_k S_k \quad T = \bigcup_k T_k \quad u = |U|$



A valid inequality is:

$$(qs+u)\sum_{i\in T} x_i + \frac{q(q-1)}{2}\sum_{i\in S\cup U} x_i \geq b$$

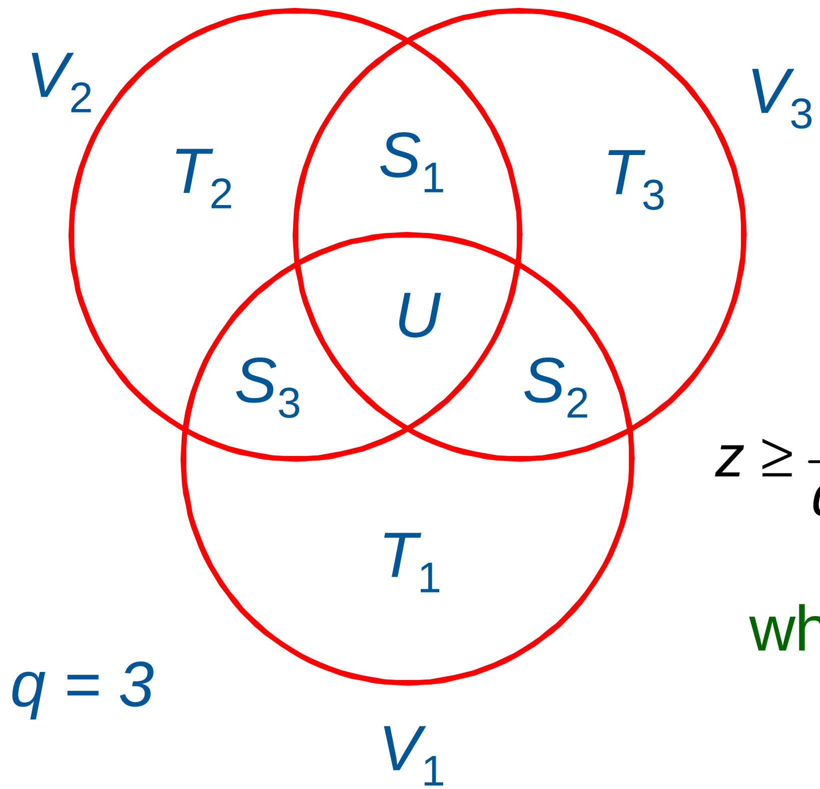
where

$$b = \frac{1}{2}q(q-1)(qs+u)(qs+u+1)$$

Properties of 0-1 mapping?

Intersecting Systems

- Facet-defining inequality. Let $S = \bigcup_k S_k \quad T = \bigcup_k T_k \quad u = |U|$



A valid bound is:

$$z \geq \frac{2}{q(q+1)} \sum_{i \in T} x_i + \frac{q-1}{(q+1)(qs+u)} \sum_{i \in S \cup U} x_i + c$$

where

$$c = \tfrac{1}{2} \frac{q-1}{q+1}(qs+u+1)$$

Benchmark Instances

with < 100 variables

Lower bound on number of colors in 0-1 model. Odd cycle cuts for $s = 1,2,3$

Instance	Odd hole	Odd cycle	Opt	Odd hole time	Odd cycle time
1-FullIns_3	2	2	3	0.4	0.4
1-FullIns_4	2	2	4	208	0.4
1-insertions_4	1.33	1.43	4	30.3	2.4
2-FullIns_3	2	2	4	0.9	0.7
2-insertions_3	1.25	1.33	3	2.9	0.2

Benchmark Instances

Lower bound on number of colors in
0-1 model. Odd cycle cuts for s = 1,2,3

Instance	Odd hole	Odd cycle	Opt	Odd hole time	Odd cycle time
3-FullIns_3	2	2	5	25.8	0.2
3-insertions_3	1.2	1.27	3	11.5	1.0
4-insertions	1.17	1.23	3	12.1	6.0
david	2	8	10	11.0	0.8
huck	2	8	10	7.2	0.3
jean	2	8	9	10.2	1.8

Benchmark Instances

Lower bound on number of colors in
0-1 model. Odd cycle cuts for s = 1,2,3

Instance	Odd hole	Odd cycle	Opt	Odd hole time	Odd cycle time
mug88_1	2	2	3	7.8	2.7
Mug88_25	2	2	3	5.3	1.7

Benchmark Instances

Lower bound on number of colors in 0-1 model. Odd cycle cuts for $s = 1,2,3$

Instance	Odd hole	Odd cycle	Opt	Odd hole time	Odd cycle time
myciel3	1.5	1.6	3	0.0	0.0
myciel4	1.5	1.6	4	0.6	0.0
myciel5	1.5	1.6	5	7.9	0.1
myciel6	1.5	1.6	3	1754	0.6

Benchmark Instances

Lower bound on number of colors in
0-1 model. Odd cycle cuts for $s = 1,2,3$

Instance	Odd hole	Odd cycle	Opt	Odd hole time	Odd cycle time
queen5_5	2	2	4	0.4	0.0
queen6_6	2	5	6	1.5	0.1
queen7_7	2	3.71	6	10.6	0.2
queen8_8	?	3.38	8	?	3.4
queen8_12	2	8	11	439.6	1.7
queen9_9	2	8	9	212.4	1.3

Future Work

- Map other known finite-domain cuts into 0-1 models. What happens?
  - Cardinality rules. Yan and JH (1999).
  - Circuit constraint (TSP). Genc-Kaya and JH (2010).
  - Cumulative constraint.
- Polyhedral analysis for other global constraints.
  - General cardinality, nvalues, sequence, regular.

Example: Circuit Constraint

- Encodes traveling salesman problem.

$$\min \sum_i c_{ix_i}$$

$$\text{circuit}(x_1,\ldots,x_n)$$

x_i = city after city i

- Completely different polyhedral structure than 0-1.
  - Classes of facets are based on position of variables in the sequence $x_1, \ldots, x_n$ rather than on subgraph structures (combs, etc.).
  - Facets can be mapped to 0-1 space.

Example: Circuit Constraint

- Cuts of order 0:

4 variables below 1st term → 4 terms (for example)

$$x_5 + x_{i_2} + x_{i_3} + x_{i_4} \geq 10$$

- Cuts of order 1:

$$x_4 + 2x_{i_2} + 2x_{i_3} + 2x_{i_4} \geq 17$$

- Cuts of order 2:

$$2x_3 + x_{i_2} + 4x_{i_3} + 4x_{i_4} \geq 25$$

$$3x_3 + 2x_{i_2} + 4x_{i_3} + 4x_{i_4} \geq 32$$

$$3x_3 + 2x_{i_2} + 5x_{i_3} + 5x_{i_4} \geq 36$$