A Filter for the Circuit Constraint

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Abstract. We present an incomplete filtering algorithm for the circuit constraint. The filter removes redundant values by eliminating nonhamiltonian edges from the associated graph. We identify nonhamiltonian edges by analyzing a smaller graph with labeled edges that is defined on a separator of the original graph. The complexity of the procedure for each separator S is approximately $O(|S|^5)$. We found that it identified all infeasible instances and eliminated about one-third of the redundant domain elements in feasible instances.

The circuit constraint can be written

$$\text{circuit}(x_1, \ldots, x_n)$$

where the domain of each x_i is $D_i \subset \{1, \ldots, n\}$. The constraint requires that $y_1, \ldots, y_n$ be a cyclic permutation of $1, \ldots, n$, where

$$\begin{aligned} &y_{i+1} = x_{y_i}, \quad i = 1, \ldots, n-1 \\ &y_1 = x_{y_n} \end{aligned}$$

Let directed graph G contain an edge (i, j) if and only if j belongs to the domain of x_i. If edge (i, j) is selected when $x_i = j$, the circuit constraint requires that the selected edges form a hamiltonian circuit or *tour* of G.

One approach to filtering the circuit constraint is to make use of necessary conditions for hamiltonicity of G. Chvátal [2, 3] analyzes several conditions, one of which (*1-toughness*) is a very restricted case of a condition we develop for filtering nonhamiltonian edges.

Some elementary techniques for filtering domains after a partial tour has been constructed are described by Shufelt and Berliner [4], whose analysis relies on the special structure of a chessboard problem, and Caseau and Laburthe [1], who solve small traveling salesman problems. Neither approach is intended for the general filtering problem in which arbitrary variable domains are given.

1 Separator Graph

Given a graph $G = (V, E)$, a set of vertices $S \subset V$ is a (vertex) *separator* of G if $V \setminus S$ induces a subgraph $\bar{G}_S$ of G with at least two connected components $C_1, \ldots, C_p$. The *separator graph* G_S for a separator S of G consists of a directed

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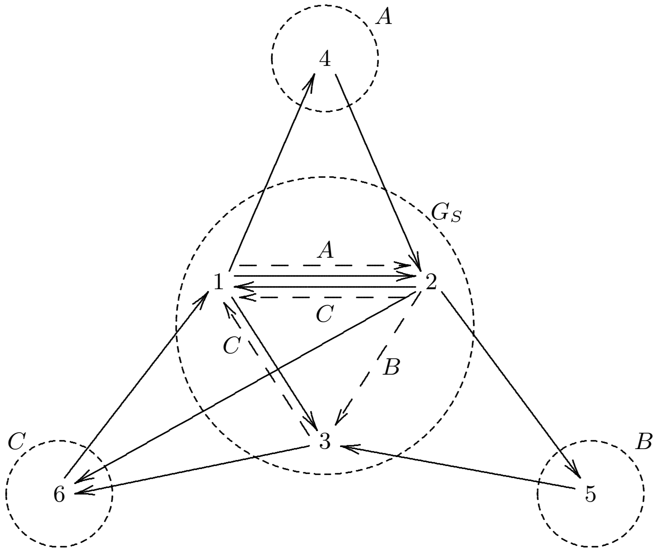

Fig. 1. Graph G on vertices $\{1, \ldots, 6\}$ contains the solid edges, and the separator graph G_S on $S = \{1, 2, 3\}$ contains the solid (unlabeled) edges and dashed (labeled) edges within the larger circle. The small circles surround connected components of the separated graph.

graph with vertex set S and edge set E_S, along with a set L_S of *labels* corresponding to the connected components of $\bar{G}_S$. E_S contains (a) an *unlabeled* edge (i, j) for each $(i, j) \in E$, as well as (b) a *labeled* edge $(i, j)^C$ whenever $C \in L_S$ and $(i, c_1), (c_2, j) \in E$ for some pair of vertices c_1, c_2 in connected component C (possibly $c_1 = c_2$ and c_1 and c_2 need not be connected by an edge).

Consider for example the graph G of Fig. 1. Vertex set $S = \{1, 2, 3\}$ separates G into three connected components that may be labeled A, B and C, each of which contains only one vertex. Thus $L_S = \{A, B, C\}$, and the separator graph G_S contains the three edges that connect its vertices in G plus four labeled edges. For example, there is an edge $(1, 2)$ labeled A, which can be denoted $(1, 2)^A$, because there is a directed path from some vertex in component A through $(1, 2)$ and back to a vertex of component A.

A hamiltonian cycle of G_S is *permissible* if it contains at least one edge bearing each label in L_S. An edge of G_S is *permissible* if it is part of some permissible hamiltonian cycle of G_S. Thus the edges $(1, 2)^A$, $(2, 3)^B$ and $(3, 1)^C$ form a permissible hamiltonian cycle in Fig. 1, and they are the only permissible edges.

Theorem 1. *If S is a separator of directed graph G, then G is hamiltonian only if G_S contains a permissible hamiltonian cycle. Furthermore, an edge of G connecting vertices in S is hamiltonian only if it is a permissible edge of G_S.*

Proof. Consider an arbitrary hamiltonian cycle H of G. We can construct a permissible hamiltonian cycle H_S for G_S as follows. Consider the sequence of vertices in H and remove those that are not in S; let $i_1, \ldots, i_m, i_1$ be the remaining sequence of vertices. H_S can be constructed on these vertices as follows. For any pair i_k, i_{k+1} (where i_{m+1} is identified with i_1), if they are adjacent in H then (i_k, i_{k+1}) is an unlabeled edge of G_S and connects i_k and i_{k+1} in H_S. If i_k, i_{k+1} are not adjacent in H then all vertices in H between i_k and i_{k+1} lie in the same connected component C of the subgraph of G induced by $V \setminus S$. This means (i_k, i_{k+1}) is an edge of G_S with label C, and $(i_k, i_{k+1})^C$ connects i_k and i_{k+1} in H_S. Since H passes through all connected components, every label must occur on some edge of H_S, and H_S is permissible.

We now show that if (i, j) with $i, j \in S$ is an edge of a hamiltonian cycle H of G, then (i, j) is an edge of a permissible hamiltonian cycle of G_S. But in this case (i, j) is an unlabeled edge of G_S, and by the above construction (i, j) is part of H_S.

Corollary 1. *If $|L_S| > |S|$ for some separator S, then G is nonhamiltonian.*

Proof. The separator graph G_S has $|S|$ vertices and therefore cannot have a hamiltonian cycle with more than $|S|$ edges.

If $|L_S| \leq |S|$ for all separators S, G is *1-tough*, adapting Chvátal's term [3] to directed graphs.

Corollary 2. *If $|L_S| = |S|$ for some separator S, then no edge connecting vertices of S is hamiltonian.*

Proof. An edge e that connects vertices in S is unlabeled in G_S. If e is hamiltonian, some hamiltonian cycle in G_S that contains e must have at least $|S|$ labeled edges. But since the cycle must have exactly $|S|$ edges, all the edges must be labeled and none can be identical to e.

2 Finding Separators

We use a straightforward breadth-first-search heuristic to find separators of G. We arrange the vertices of G in levels as follows. Arbitrarily select a vertex i of G as a *seed* and let level 0 contain i alone. Let level 1 contain all neighbors of i in G. Let level k (for $k \geq 2$) contain all vertices j of G such that (a) j is a neighbor of some vertex on level $k-1$, and (b) j does not occur in levels 0 through $k-1$. If maximum level $m \geq 2$, the vertices on any given level k ($0 < k < m$) form a separator of G. Thus the heuristic yields $m-1$ separators.

The heuristic can be run several times as desired, each time beginning with a different vertex on level 0.

3 Cardinality Filter and Vertex Degree Filtering

The next step of the algorithm is to identify nonpermissible edges of G_S for each separator S by a relaxation of the permissibility condition.

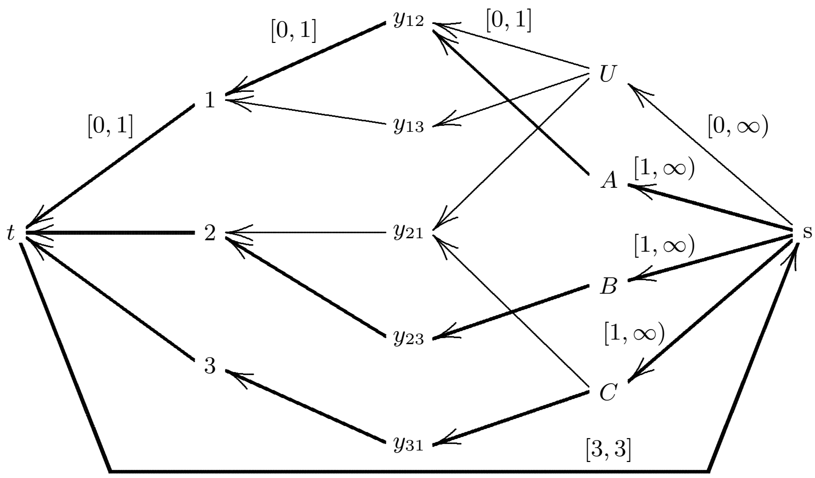

Fig. 2. Flow model for simultaneous gcc and out-degree filtering of nonhamiltonian edges. Heavy lines show the only feasible flow.

Let Y contain a variable y_{ij} for each *ordered* pair of vertices i, j in G_S. The domain of y_{ij} contains label C for each edge $(i,j)^C$ in G_S and the element U if unlabeled edge (i,j) is in G_S. A permissible hamiltonian cycle must satisfy the constraint

$$\text{gcc}(Y, (C_1, \ldots, C_p, U), (1, \ldots, 1, 0), (\infty, \ldots, \infty)) \tag{1}$$

Vertex degree filtering is based on the fact that the in-degree and out-degree of every vertex in a hamiltonian cycle is one. Constraint (1) can be combined with out-degree filtering by constructing a capacitated flow graph G_S^{out} with the following vertices

source s and sink t
U and $C_1, \ldots, C_p$
a vertex for each $y_{ij} \in Y$
a vertex for every vertex of G_S

and the following directed edges

(s, C_i) with capacity range $[1, \infty)$ for $i = 1, \ldots, p$
(s, U) with capacity range $[0, \infty)$
(C, y_{ij}) with capacity range $[0, 1]$ for every edge $(i,j)^C \in E_S$
(U, y_{ij}) with capacity range $[0, 1]$ for every unlabeled edge $(i,j) \in E_S$
(y_{ij}, i) with capacity range $[0, 1]$ for every ordered pair (i, j) such that (i, j) or $(i, j)^C$ belongs to E_S
(i, t) with capapcity range $[0, 1]$ for every vertex i of G_S
return edge (t, s) with capacity range $[|S|, |S|]$

The gcc and out-degree constraints are simultaneously satisfiable if and only if G_S^{out} has a feasible flow. The same is true of the graph G_S^{in} constructed in an analogous way to enforce in-degree constraints. Thus

Let G be the directed graph associated with circuit$(x_1, \ldots, x_n)$.
Let D_i be the current domain of x_i for $i = 1, \ldots, n$.
Let s be a limit on the size of separators considered.
For one or more vertices i of G:
  Use the breadth-first-search heuristic to create a collection $\mathcal{S}$ of separators, with i as the seed.
  For each $S \in \mathcal{S}$ with $|S| \leq s$:
    For $G'_S = G_S^{out}, G_S^{in}$:
      If G'_S has a feasible flow f then:
        For each edge (U, y_{ij}) of G'_S on which f places zero flow:
          If there is no augmenting path from y_{ij} to U then delete j from D_i.
      Else stop; circuit$(x_1, \ldots, x_n)$ is infeasible.

Fig. 3. Filtering algorithm for the circuit constraint

Theorem 2. *An edge (i, j) of G is nonhamiltonian if there is a separator S of G for which the maximum flow on arc (U, y_{ij}) of either G_S^{out} or G_S^{in} is zero.*

This can be checked by first computing a feasible flow f on G_S^{out} and on G_S^{in}. The maximum flow on (U, y_{ij}) is zero if (a) f places zero flow on (U, y_{ij}), and (b) there is no augmenting path from y_{ij} to U.

For example, the network G_S^{out} for the graph G and separator S of Fig. 1 is shown in Fig. 2. Since the flow of zero on edges (U, y_{12}), (U, y_{13}) and (U, y_{21}) is maximum in each case, the three edges $(1, 2)$, $(1, 3)$, and $(2, 1)$ are nonhamiltonian.

4 The Algorithm and Computational Results

The filtering algorithm (Fig. 3) has complexity of approximately $O(|S|^5)$ for each seoparator S. In preliminary computational tests on several thousand random graphs with up to 15 vertices, we detected all nonhamiltonian graphs and eliminated about 1/3 of nonhamiltonian edges in hamiltonian graphs.

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