Finite Domain Cuts for Minimum Bandwidth

J. N. Hooker
Carnegie Mellon University, USA

Joint work with
Alexandre Friere, Cid de Souza
State University of Campinas, Brazil

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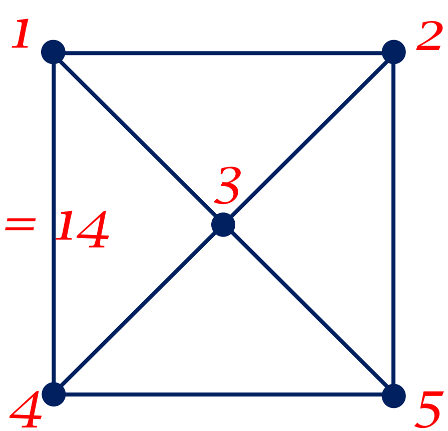
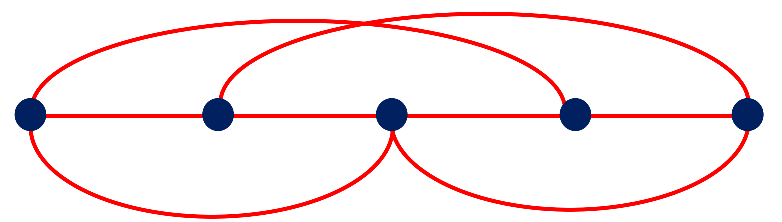
Two Related Problems

- They differ only in the objective function.
- Minimum Linear Arrangement problem:
  - Label vertices of a graph to minimize **sum** of edge lengths.
  - Application: minimize wiring on a linear circuit board

Min edge sum = 14





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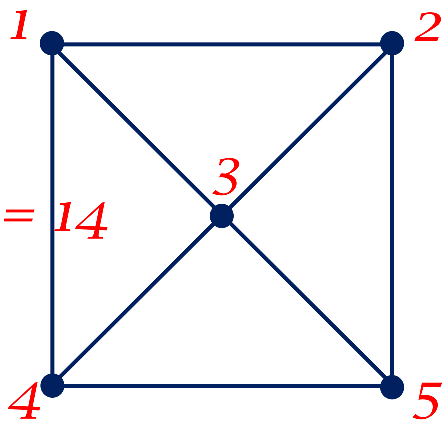


Min edge sum = 14

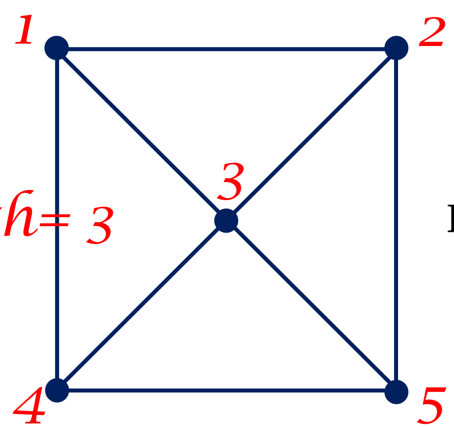
$$\min \left\{ \sum_{(i,j)\in E} u_{ij} \;\middle|\; \begin{array}{c} u_{ij} = \left|\pi_i - \pi_j\right| \\ \pi \text{ is perm. of } 1,\ldots,n \end{array} \right\}$$

Two Related Problems

- They differ only in the objective function.

- Minimum Bandwidth problem:
  - Label vertices of a graph to minimize **maximum** edge length.
  - Application: sparse matrix calculation
  - Application: ordering variables to minimize backtracking

Min bandwidth = 3



$$\min \left\{ \max_{(i,j)\in E} \{u_{ij}\} \;\middle|\; \begin{array}{c} u_{ij} = |\pi_i - \pi_j| \\ \pi \text{ is perm. of } 1,\ldots,n \end{array} \right\}$$

Finite-Domain Cuts

- Difficult to find good bounds, polyhedral or otherwise.
  - 0-1 models have weak relaxations.

- Idea: Use finite domain cuts
  - Formulate problem with **finite domain** variables rather than 0-1 variables.
  - Find valid inequalities.
  - This was useful for **graph coloring.**
    - Bergman and Hooker, CPAIOR 2012.
    - Let x_i = color assigned to vertex *i.*
    - Obtained tighter bounds in less time.

Finite-Domain Cuts

- For linear arrangement and bandwidth, use edge length variables u_{ij} only.
  - Unclear how to give problem a complete inequality formulation.
  - But we can derive valid cuts.

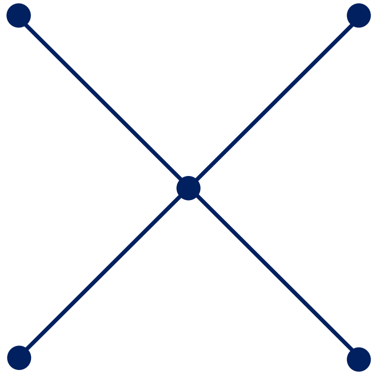
$$\min\left\{ \sum_{(i,j)\in E} u_{ij} \;\middle|\; \text{valid cuts in } u_{ij} \right\}$$

$$\min\left\{ w = \max_{(i,j)\in E}\left\{u_{ij}\right\} \;\middle|\; \text{valid cuts in } w, u_{ij} \right\}$$

Min Linear Arrangement

- This idea already applied to MLA problem.
  - Substantial improvement over existing bounds.
    - Caprara, Letchford, Salazar-González, *IJOC* 2011.
  - Polytope hard to analyze.
    - CLS studied **dominant** of polytope instead.
    - Facets found for **stars**, etc.

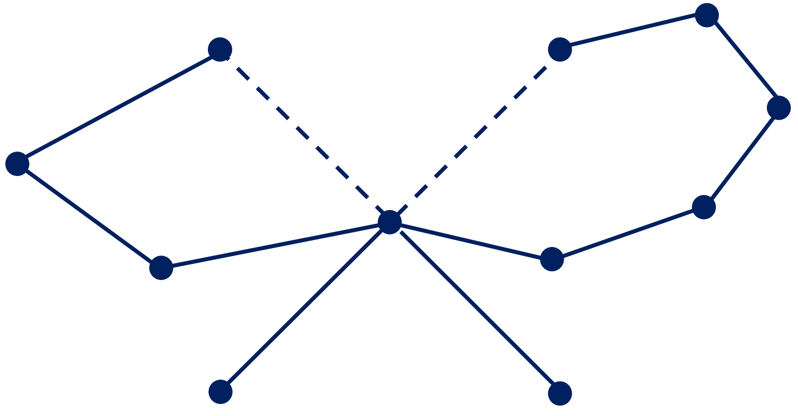
$k = 5$:



$$\sum_{(i,j)} u_{ij} \geq \left\lfloor \frac{k^2}{4} \right\rfloor$$

Min Linear Arrangement

- This idea has already been applied to MLA problem.
  - CLS generated cuts for stars in **complete graph**.
    - Filled in **missing edges** with **paths** in original graph.
    - Resulting cuts remain valid (not facet defining).
    - Separation algorithm found the paths.
    - Stars did most of the work.



Min Linear Arrangement

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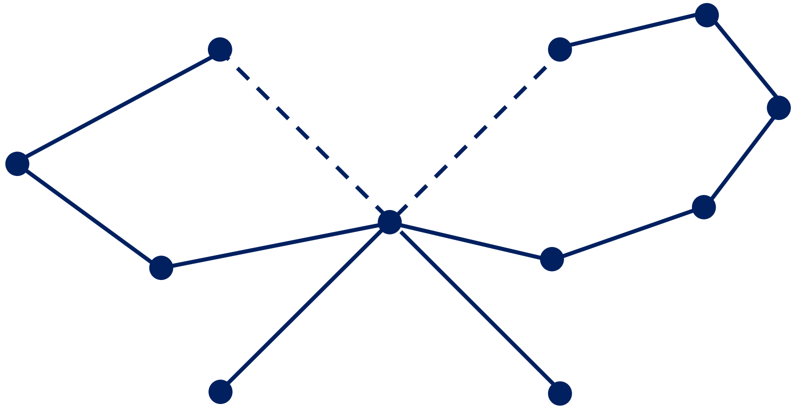
  - CLS generated cuts for stars in **complete graph**.
    - Filled in **missing edges** with **paths** in original graph.
    - Resulting cuts remain valid (not facet defining).
    - Separation algorithm found the paths.
    - Stars did most of the work.
  - So far, our cuts do not improve on stars + separation.



Min Bandwidth

- Missing edges trick does not work for min bandwidth.
  - Resulting cuts are not valid.

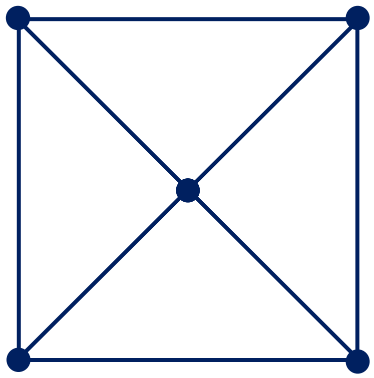
- We experiment with alternate approaches to finding cuts.
  - Focus on validity, not facets.
  - Identify cuts for large structures that fill much of the graph.
  - Modify cuts to restore validity after modifying structures.
  - Use counting arguments.

Known Bounds

- Density bound is best-known bound.
    - Chvátal (1970)
  - NP-hard to compute.
  - Not a polyhedral cut.

$$w \geq \max_{S \subseteq V} \left\lceil \frac{|S|-1}{d(S)} \right\rceil$$

diameter of S



$$w \geq \left\lceil \frac{5-1}{2} \right\rceil = 2$$

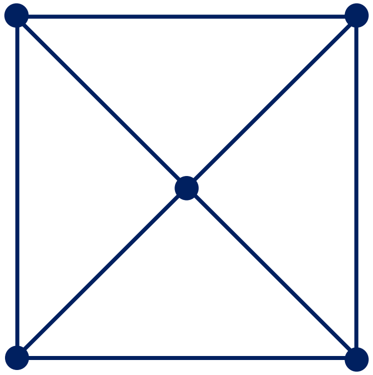
bandwidth = 3

Known Bounds

- Alternate bound (call it CS bound)
    - Caprara, Salazar-González, *IJOC* 2005.
  - P-time to compute.
  - But not a polyhedral cut.

$$w \geq \min_{v \in V} \max_{S : v \in S} \left\lceil \frac{|S| - 1}{d(v, S)} \right\rceil$$

max distance from v to vertex in S



$$w \geq \min\left\{ \left\lceil \frac{4-1}{1} \right\rceil, \left\lceil \frac{5-1}{1} \right\rceil \right\} = 3$$

bandwidth = 3

Our Proof Strategy for Bounds

- Find an upper bound on edge sum in a structured subgraph, as a function of bandwidth *w*:

$$\sum_{(i,j)} u_{ij} \leq U(w)$$

  - Then solve for bound on *w*: $w \geq U^{-1}\left(\sum_{(i,j)} u_{ij}\right)$

- Combine with lower bounds on edge sum:

$$\sum_{(i,j)} u_{ij} \geq L(w)$$

  - and with lower bounds due to CLS.

Path Cuts

- Path on *k* vertices has max edge sum

$$\sum_{(i,j)} u_{ij} \le U(w) = (k-1)w$$

- So we have the cut $w \ge U^{-1}\left(\sum_{(i,j)} u_{ij}\right) = \frac{1}{k-1}\sum_{(i,j)} u_{ij}$
  - There are many long paths, but the cut is very weak.

$$w \ge \tfrac{1}{5}\sum_{(i,j)} u_{ij}$$

For w = 5: *0* *5* *10* *15*
20 *25*

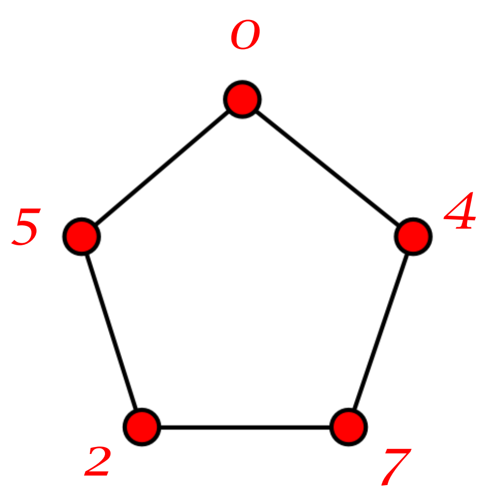
Cycle Cuts

- *k*-cycle has max edge sum

$$\sum_{(i,j)} u_{ij} \leq kw - 2 \quad \text{if } k \text{ is even}$$

$$\sum_{(i,j)} u_{ij} \leq (k-1)w \quad \text{if } k \text{ is odd}$$

For $w = 5$:



$$\sum_{(i,j)} u_{ij} \leq 4w$$

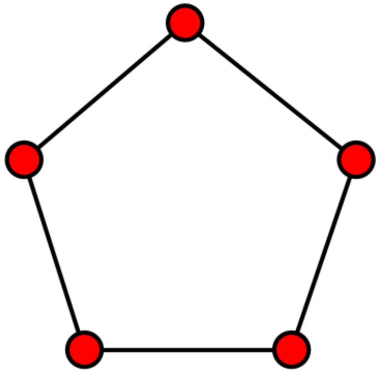
Cycle Cuts

- This yields the (sharp) bound

$$w \geq \frac{1}{k}\sum_{(i,j)} u_{ij} + \frac{2}{k} \quad \text{if } k \text{ is even}$$

$$w \geq \frac{1}{k-1}\sum_{(i,j)} u_{ij} \quad \text{if } k \text{ is odd}$$

  - Slightly stronger than for path



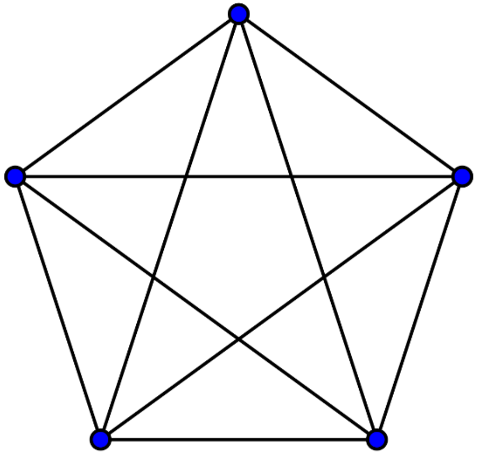
$$w \geq \tfrac{1}{4}\sum_{(i,j)} u_{ij}$$

Clique Cuts

- Clique of size *k* has max edge sum

$$\sum_{(i,j)} u_{ij} \le \frac{k^2}{4} w - \frac{k}{12}(k-1)(k-2) \qquad \text{if } k \text{ is even}$$

$$\sum_{(i,j)} u_{ij} \le \frac{k^2-1}{4} w - \frac{k+1}{12}(k-1)(k-3) \qquad \text{if } k \text{ is odd}$$



$$\sum_{(i,j)} u_{ij} \le 6w - 4$$

Clique Cuts

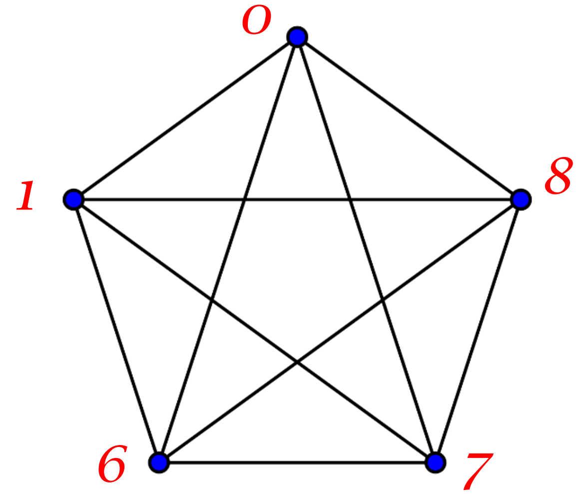
- Clique of size *k* has max edge sum

$$\sum_{(i,j)} u_{ij} \le \frac{k^2}{4} w - \frac{k}{12}(k-1)(k-2) \qquad \text{if } k \text{ is even}$$

$$\sum_{(i,j)} u_{ij} \le \frac{k^2-1}{4} w - \frac{k+1}{12}(k-1)(k-3) \qquad \text{if } k \text{ is odd}$$

  – Achieve this max by using *k*/2 smallest and *k*/2 largest vertex labels in {0, …, *w*} for *k* even, similarly for *k* odd.

For $w = 8$:



$$\sum_{(i,j)} u_{ij} \le 6w - 4$$

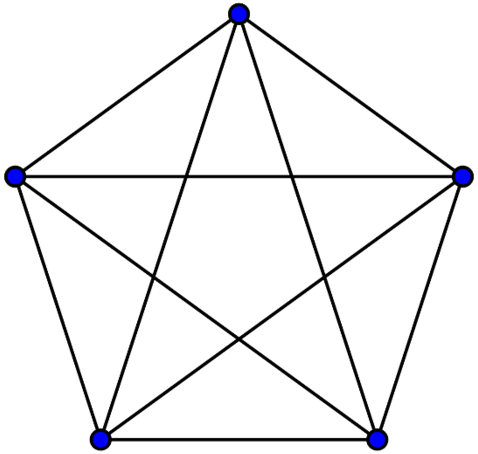
Clique Cuts

- This yields the (sharp) bound:

$$w \geq \frac{4}{k^2} \sum_{(i,j)} u_{ij} + \frac{1}{3k}(k-1)(k-2) \quad \text{if } k \text{ is even}$$

$$w \geq \frac{4}{(k^2-1)} \sum_{(i,j)} u_{ij} + \frac{1}{3}(k-3) \quad \text{if } k \text{ is odd}$$

  - Strong cut, but hard to find large cliques



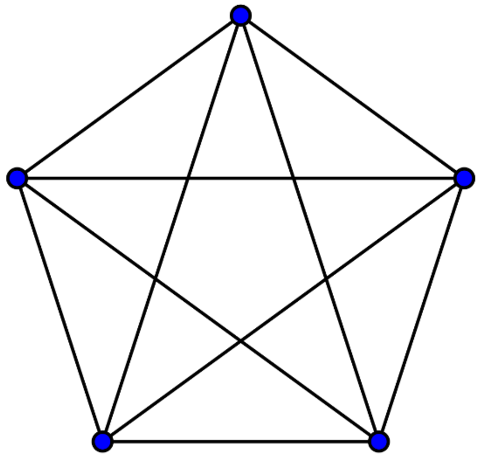
$$w \geq \tfrac{1}{6} \sum_{(i,j)} u_{ij} + \tfrac{2}{3}$$

Clique Cuts

- We also have the (sharp) lower bound on edge sum

$$\sum_{(i,j)} u_{ij} \geq \frac{k}{6}(k+1)(k-1) = \binom{k}{3}$$

  - Combined with LB on *w*, this yields $w \geq k-1$ which is sharp for a clique and same as density and CS bound.



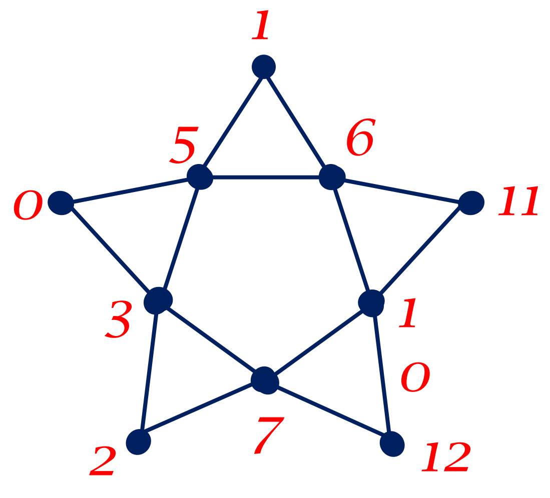
$$w \geq 4$$

Clique Cycles

- Cycles of cliques can provide good bounds for large structures.
- Cycle of *k* 3-cliques has max edge sum

$$\sum_{(i,j)} u_{ij} \leq 2kw$$

For $w = 5$:



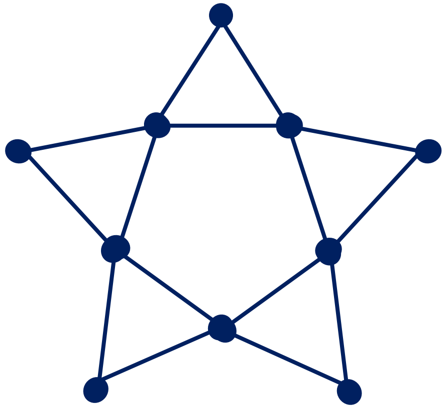
$$\sum_{(i,j)} u_{ij} \leq 10w$$

Clique Cycles

- This yields the (sharp) bound

$$w \geq \frac{1}{2k} \sum_{(i,j)} u_{ij}$$

  - For odd and even *k*



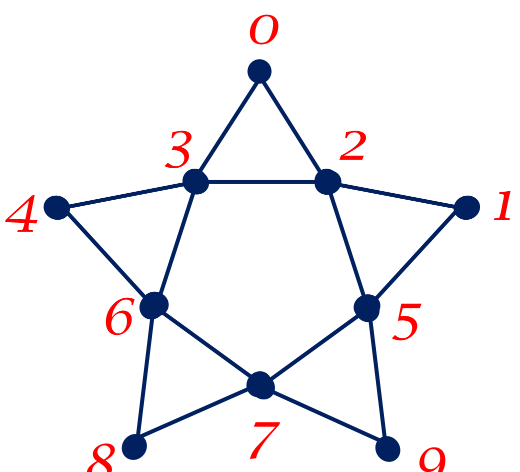
$$w \geq \frac{1}{10} \sum_{(i,j)} u_{ij}$$

Clique Cycles

- We also have the (sharp) lower bound on edge sum

$$\sum_{(i,j)} u_{ij} \geq 8(k-1)$$

  - Combined with LB on *w*, this yields $w \geq \left\lceil 4 - \frac{4}{k} \right\rceil$
  - For *k* = 5, this yields *w* ≥ 4 (sharp), stronger than density bound 3, same as CS bound.

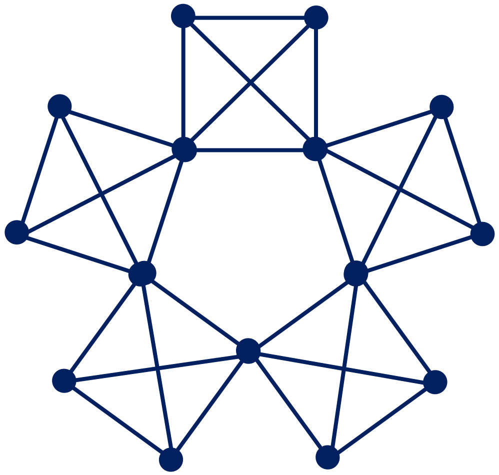


$$\sum_{(i,j)} u_{ij} \geq 32$$

Clique Cycles

- Conjecture: Cycle of *k* 4-cliques has max edge sum

$$\sum_{(i,j)} u_{ij} \leq 4kw - 2k + 1$$

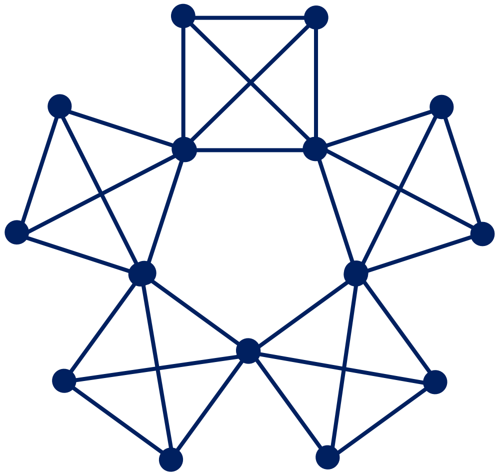


$$\sum_{(i,j)} u_{ij} \leq 20w - 9$$

Clique Cycles

- This yields the bound

$$w \geq \frac{1}{4k} \sum_{(i,j)} u_{ij} + \frac{2k-1}{4k}$$



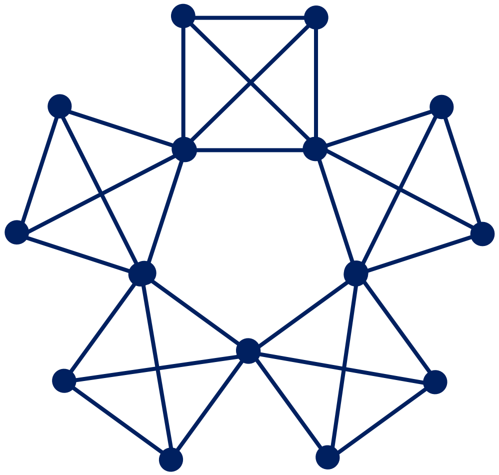
$$w \geq \tfrac{1}{20} \sum_{(i,j)} u_{ij} + \tfrac{9}{20}$$

Clique Cycles

- We also have the (sharp) lower bound on edge sum

$$\sum_{(i,j)} u_{ij} \geq 16k - 15$$

  - Combined with LB on *w*, this yields $w \geq \left\lceil 4.5 - \frac{1}{4k} \right\rceil$
  - For $k = 5$, this yields $w \geq 5$ (sharp). Density and CS bound are also 5.

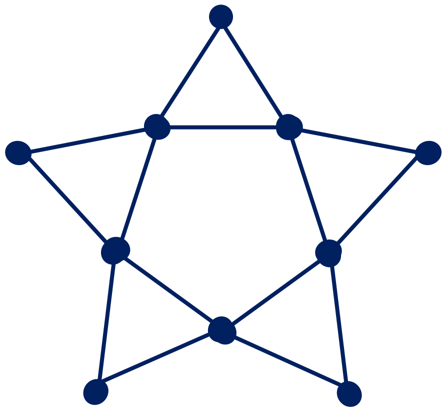


$$\sum_{(i,j)} u_{ij} \geq 65$$

Cuts when Edges Are Missing

- A certificate of a sharp bound on edge sum can yield a sharp bound when edges are removed.
  - Start with a sharp upper bound *U*(*w*) on edge sum**.**

For $w = 5$:

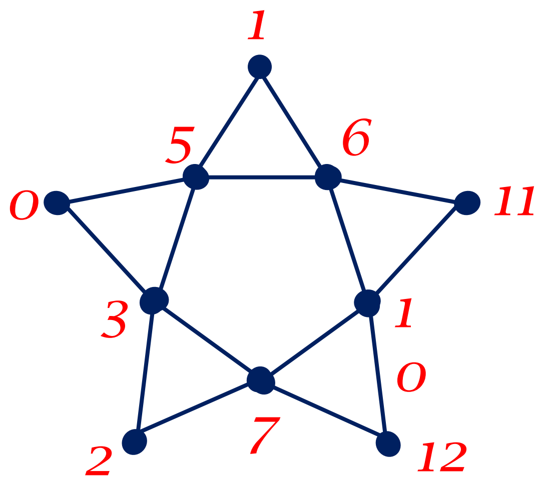


$$\sum_{(i,j)} u_{ij} \le 10w$$

Cuts when Edges Are Missing

- A certificate of a sharp bound on edge sum can yield a sharp bound when edges are removed.
  - Start with a sharp upper bound $U(w)$ on edge sum.
  - Let v be a vertex labeling that achieves edge sum $U(w)$.

For $w = 5$:

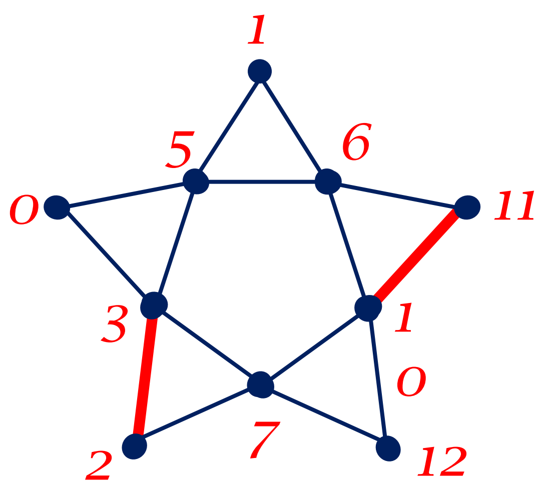


$$\sum_{(i,j)} u_{ij} \leq 10w$$

Cuts when Edges Are Missing

- A certificate of a sharp bound on edge sum can yield a sharp bound when edges are removed.
  - Start with a sharp upper bound $U(w)$ on edge sum.
  - Let v be a vertex labeling that achieves edge sum $U(w)$.
  - Remove m edges (i, j) for which $|v_i - v_j| = 1$.

For $w = 5$:

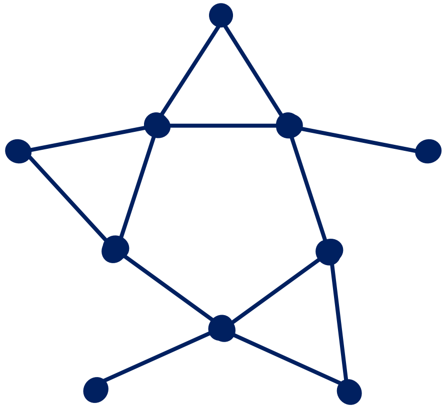


$$\sum_{(i,j)} u_{ij} \leq 10w$$

Cuts when Edges Are Missing

- A certificate of sharp bound on edge sum can yield a sharp bound when edges are removed.
  - Start with a sharp upper bound $U(w)$ on edge sum.
  - Let v be a vertex labeling that achieves edge sum $U(w)$.
  - Remove m edges (i, j) for which $|v_i - v_j| = 1$.
  - Now $U(w) - m$ is a sharp upper bound on edge sum.

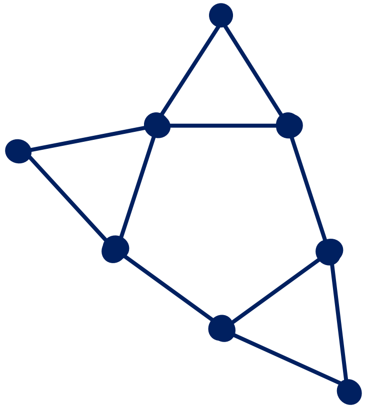
For $w = 5$:



$$\sum_{(i,j)} u_{ij} \leq 10w - 2$$

Cuts when Edges Are Missing

- Can get a valid (nonsharp) bound after removing any *m* edges
  - $U(w) - m$ is an upper bound on edge sum.



$$\sum_{(i,j)} u_{ij} \le 10w - 4$$

Counting Arguments

- Take any graph with 12 vertices.
  - And any subgraph with *m* edges.
  - For any bandwidth *w*, compute maximum edge sum.
  - Sum the first *m* edge labels starting with *w*.

Edge Label	**Max No. Edges**	**Possible edges**
11	1	(12,1)
10	2	(12,2) (11,1)
9	3	(12,3) (11,2) (10,1)
8	4	(12,4) (11,3) (10,2) (9,1)
7	5	(12,5) (11,4) (10,3) (9,2) (8,1)
6	6	(12,6) (11,5) (10,4) (9,3) (8,2) (7,1)
etc.		

Counting Arguments

- For example, assuming subgraph has 7 edges
  - For bandwidth $z = 11$
  - Max edge sum is $1 \cdot 11 + 2 \cdot 10 + 3 \cdot 9 + 1 \cdot 8 = 66$

Edge Label	Max No. Edges	Possible edges
11	1	**(12,1)**
10	2	**(12,2) (11,1)**
9	3	**(12,3) (11,2) (10,1)**
8	4	**(12,4)** (11,3) (10,2) (9,1)
7	5	(12,5) (11,4) (10,3) (9,2) (8,1)
6	6	(12,6) (11,5) (10,4) (9,3) (8,2) (7,1)
etc.		

Counting Arguments

- For example, assuming subgraph has 7 edges
  - For bandwidth *z* = 10
  - Max edge sum is 2·10 + 3·9 + 2·8 = 63

Edge Label	Max No. Edges	Possible edges
11	1	(12,1)
→ **10**	2	**(12,2) (11,1)**
9	3	**(12,3) (11,2) (10,1)**
8	4	**(12,4) (11,3)** (10,2) (9,1)
7	5	(12,5) (11,4) (10,3) (9,2) (8,1)
6	6	(12,6) (11,5) (10,4) (9,3) (8,2) (7,1)
etc.		

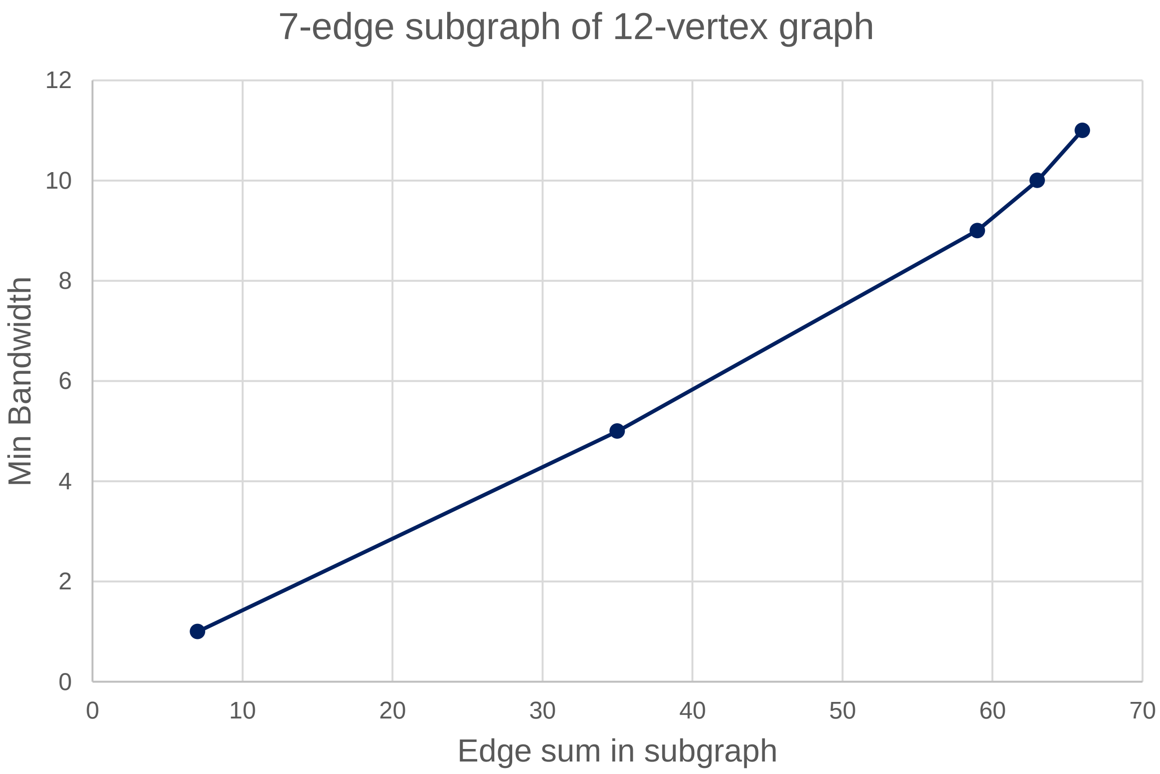
Counting Arguments

- For example, assuming subgraph has 7 edges
  - For bandwidth $z = 9$
  - Max edge sum is $3 \cdot 9 + 4 \cdot 8 = 59$

Edge Label	Max No. Edges	Possible edges
11	1	(12,1)
10	2	(12,2) (11,1)
9	3	**(12,3) (11,2) (10,1)**
8	4	**(12,4) (11,3) (10,2) (9,1)**
7	5	(12,5) (11,4) (10,3) (9,2) (8,1)
6	6	(12,6) (11,5) (10,4) (9,3) (8,2) (7,1)
etc.		

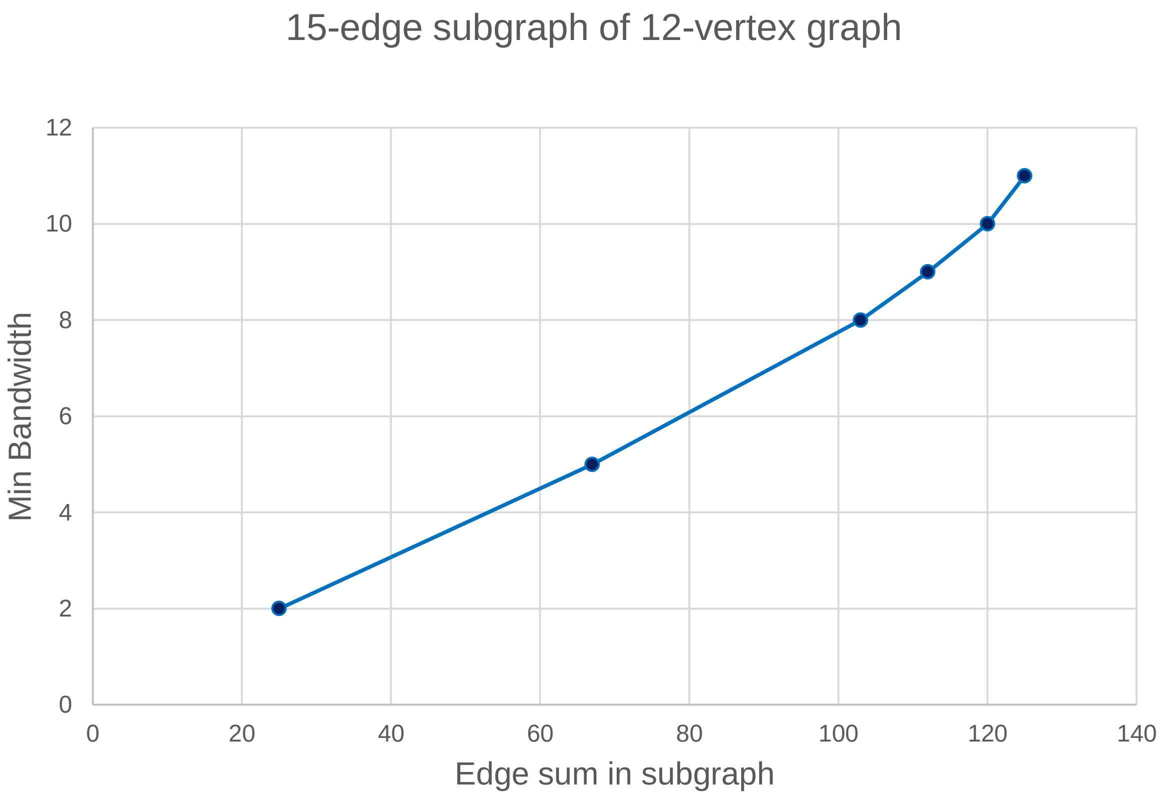
Counting Arguments

- This yields family of **valid cuts**
  - of the form $w \geq \alpha \sum_{(i,j)} u_{ij} + \beta$



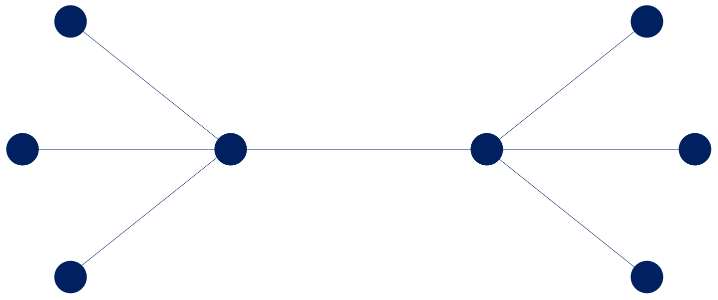
Counting Arguments

- This yields family of **valid cuts**
  - of the form $w \geq \alpha \sum_{(i,j)} u_{ij} + \beta$



Counting Arguments

- For stronger cuts, we must consider structure of subgraph.
  - We will use degree constraints.
- Example:
  - Suppose $n = 12$.
  - Consider subgraph with $m = 7$.
  - Enforce degree constraints
    - 2 vertices degree 4
    - 6 vertices degree 1



Counting Arguments

- Use greedy algorithm to get upper bound on edge sum
  - For bandwidth $z = 10$
  - UB is $1 \cdot 11 + 2 \cdot 10 + 2 \cdot 9 + 2 \cdot 8 = 65$ (not 66)
    - Max is 65

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→ **11**	1	**(12,1)**
10	2	**(12,2) (11,1)**
9	3	**(12,3)** (11,2) **(10,1)**
8	4	**(12,4)** (11,3) (10,2) **(9,1)**
7	5	(12,5) (11,4) (10,3) (9,2) (8,1)
6	6	(12,6) (11,5) (10,4) (9,3) (8,2) (7,1)
etc.		

Counting Arguments

- Use greedy algorithm to get upper bound on edge sum
  - For bandwidth $z = 10$
  - Max edge sum is $2 \cdot 10 + 2 \cdot 9 + 2 \cdot 8 + 1 \cdot 7 = 61$ (not 63)
    - Max is 58

	Edge Label	Max No. Edges	Possible edges
	11	1	(12,1)
→	**10**	2	**(12,2) (11,1)**
	9	3	**(12,3)** (11,2) **(10,1)**
	8	4	**(12,4)** (11,3) (10,2) **(9,1)**
	7	5	**(12,5)** (11,4) (10,3) (9,2) (8,1)
	6	6	(12,6) (11,5) (10,4) (9,3) (8,2) (7,1)
	etc.		

Counting Arguments

- Research question:
  - For what structures does greedy algorithm yield valid upper bound?

Edge Label	Max No. Edges	Possible edges
11	1	(12,1)
10	2	**(12,2) (11,1)**
9	3	**(12,3)** (11,2) **(10,1)**
8	4	**(12,4)** (11,3) (10,2) **(9,1)**
7	5	**(12,5)** (11,4) (10,3) (9,2) (8,1)
6	6	(12,6) (11,5) (10,4) (9,3) (8,2) (7,1)
etc.		

Ongoing Research

- Explore other substructures with dense regions.
- Develop cuts based on certificates.
- Identify structures for which greedy algorithm works in counting argument.
- Find separation algorithms.
- Computational tests.