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Graph Coloring Inequalities from All-different Systems

David Bergman · J. N. Hooker

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Abstract We explore the idea of obtaining valid inequalities for a 0–1 model from a finite-domain constraint programming formulation of the problem. In particular, we formulate a graph coloring problem as a system of all-different constraints. By analyzing the polyhedral structure of all-different systems, we obtain facet-defining inequalities that can be mapped to valid cuts in the classical 0–1 model of the problem. We focus on cuts corresponding to cycles and webs and show that they are stronger than known cuts for these structures. We also identify path cuts and show they do not strengthen the bound. Computational experiments for a set of benchmark instances reveal that finite-domain cycle cuts often deliver tighter bounds, in less time, than classical 0–1 cuts.

Keywords graph coloring, finite-domain variables, facet-defining inequalities

1 Introduction

The vertex coloring problem is one of the best known optimization problems defined on a graph. It asks how many colors are necessary to color the vertices so that adjacent vertices receive different colors. The minimum number of colors is the chromatic number of the graph.

The problem can be given a 0–1 programming model or a constraint programming (CP) model. The 0–1 model benefits from several known classes of facet-defining inequalities that tighten its continuous relaxation. The CP model consists of all-different constraints and is normally solved without the help of a continuous relaxation.

Nonetheless, facet-defining inequalities can be derived for the CP model as well as for the 0–1 model, if its finite-domain variables are interpreted as having

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David Bergman
School of Business, University of Connecticut, E-mail: david.bergman@business.uconn.edu

J. N. Hooker
Tepper School of Business, Carnegie Mellon University, E-mail: jh38@andrew.cmu.edu

numerical values. These inequalities can be mapped into the 0–1 model, using a simple change of variable, to obtain valid cuts that we call *finite-domain cuts*. Because the CP model has a very different polyhedral structure than the 0–1 model, one might expect the finite-domain cuts to be different from known 0–1 cuts. We find that at least some families of finite-domain cuts, corresponding to cyclic structures and webs, are not only different from known cuts associated with cycles [4, 13, 14, 16] but provide tighter bounds.

This is an instance of a general strategy: reformulate a given 0–1 model in terms of finite-domain variables, study the resulting polyhedron, and map any cuts back into the 0–1 model. Binary variables frequently encode a choice that might just as well be encoded by a single finite-domain variable. For example, a 0–1 variable y_{ij} might represent whether job j is assigned to worker i, whether task i begins at time j, or whether stop j follows stop i on a bus route. These choices can be represented by a finite-domain variable x_i that indicates which job is assigned worker i, at what time task i starts, or which stop follows stop i.

The polyhedral structure of some finite-domain CP models, including all-different systems, has been studied. Yet the strength of the resulting cuts has not been directly compared with that of cuts in a 0–1 model. Linear inequalities derived for a CP model generally remain linear when mapped into a 0–1 model. This allows finite-domain cuts to be combined with 0–1 cuts that may have complementary strengths. To our knowledge, such a strategy has not previously been examined.

We present here our results for webs, odd cycles, and paths. All bounds mentioned are bounds on the chromatic number in the vertex coloring problem.

– Finite-domain *web cuts*, when mapped into the 0–1 model, yield provably tighter bounds for webs than standard web cuts.
– *Odd cycles* are a generalization of odd holes. We show that in the special case of odd holes, finite-domain cuts yield provably tighter bounds than standard odd-hole and clique cuts. In the general case of odd cycles, only two finite-domain cuts for a given cycle provide a provably tighter bound than hundreds or thousands of odd-hole and clique cuts that can be generated for that cycle.
– By contrast, we identify finite-domain *path cuts* that do not improve existing bounds. When mapped into 0–1 space, they have no effect on the bound provided by the standard 0–1 model.

We make no claim that graph coloring problems are most efficiently solved using a purely polyhedral approach, although there have been efforts in this direction [13, 14]. Rather, we claim that if relaxation bounds play a role in the solution method, finite-domain cuts can provide tighter bounds than standard 0–1 cuts.

As it happens, the graph coloring problem has a linear objective function in both the CP and the 0–1 models. Odd-cycle cuts can therefore be added directly to an LP relaxation of the CP model, if desired. This allows one to obtain a bound by solving this relaxation rather than an LP relaxation of the 0–1 model, and we find that it is in fact the same bound. If other families of finite-domain cuts are developed, this suggests the possibility of obtaining bounds from a relaxation of the CP model rather than from the much larger 0–1 relaxation.

We begin below with a problem statement and a brief literature review. We then describe the mapping of finite-domain cuts into the 0–1 space and prove some of its elementary properties. We next derive facet-defining inequalities for odd cycles, webs, and paths, and study their properties when mapped into the 0–1 space. In addition, we show that each of these inequalities gives rise to a related inequality that bounds the chromatic number, a fact that is essential to obtaining good results. A section on computational results compares the strength of finite-domain cuts and known 0–1 cuts on odd cycles and webs. It also demonstrates the advantages of odd-cycle cuts on a set of benchmark instances from the DIMACS library. The paper concludes with a summary and suggestions for future research.

2 The Problem

Given an undirected graph G with vertex set V and edge set E, the vertex coloring problem is to assign a color x_i to each vertex $i \in V$ so that $x_i \neq x_j$ for each $(i, j) \in E$. We seek a solution with the minimum number of colors; that is, a solution that minimizes $|\{x_i \mid i \in V\}|$.

The vertex coloring problem can be formulated as a system of all-different constraints. An all-different constraint alldiff(X) requires that the variables in set X take pairwise distinct values. Let $\{V_k \mid k \in K\}$ be the vertex sets of the maximal cliques of G, and let X_k be the set of variables x_i with $i \in V_k$. Let the colors be denoted by distinct nonnegative numbers v_j for $j \in J$, so that each variable x_i has the finite domain $D = \{v_j \mid j \in J\}$. Then the problem of minimizing the number of colors is

$$\begin{aligned} &\min\ z \\ &z \geq x_i,\ i \in V \\ &\text{alldiff}(X_k),\ k = 1, \ldots, K \\ &x_i \in D = \{v_j \mid j \in J\},\ i \in V \end{aligned} \tag{1}$$

Note that minimizing $\sum_{i \in V} x_i$ does not result in the vertex coloring problem, because an optimal coloring may not minimize $\sum_{i \in V} x_i$.

Any clique cover $\{V_k \mid k \in K\}$ suffices to formulate the coloring problem. However, in the analysis to follow, we assume that the 0–1 model for a graph that consists of a cycle, web, or path is based on covers consisting of maximal cliques. A clique cover for more general instances can be obtained with a heuristic algorithm, as indicated in Section 7.

It is convenient to assume that $|V| = n$ colors $v_0, \ldots, v_{n-1}$ are available. We also assume $v_0 < \cdots < v_{n-1}$. An initial question is how to select numerical domain values $v_0, \ldots, v_{n-1}$, and how polyhedral structure depends on the selection. We note that this same question arises in 0–1 programming, because the numerical domain of a boolean variable need not be $\{0, 1\}$. In the boolean case, polyhedral results are valid for any binary domain, after appropriate adjustments in the coefficients and right-hand sides of valid inequalities.

We will refer to a valid inequality for the convex-hull of feasible solutions to (1) defined only on x variables an *x-cut* and a valid inequality defined on the x and z variables a *z-cut.*

The issue is more complicated for general finite domains, but we find that the x-cuts identified here are valid for arbitrary nonnegative domain values, while z-cuts are valid for any domain of the form $D_\delta = \{0, \delta, 2\delta \ldots, (n-1)\delta\}$, where $\delta > 0$. In practice, it is convenient to use domain D_1, because in this case the minimum color number z is one less than the chromatic number.

A standard 0–1 model for the coloring problem uses binary variables y_{ij} to denote whether vertex i receives color j, and binary variables w_j that indicate whether color j is used. The model is

$$\begin{aligned}
&\min \sum_{j\in J} w_j \\
&\sum_{j\in J} y_{ij} = 1,\;\; i \in V && (a) \\
&\sum_{i\in V_k} y_{ij} \le w_j,\;\; j\in J,\; k\in K && (b) \\
&y_{ij} \in \{0,1\},\;\; i\in V,\; j\in J \\
&w_j \in \{0,1\},\;\; j\in J
\end{aligned} \tag{2}$$

We let P_X be the convex hull of feasible solutions in the finite-domain model (1) and P_Y be the convex hull of feasible solutions in the 0–1 model (2).

The finite-domain variables x_i are readily expressed in terms of the 0–1 variables y_{ij}:

$$x_i = \sum_{j\in J} v_j y_{ij} \tag{3}$$

This allows any valid inequality for model (1) to me mapped to a valid inequality for (2) by substituting the expression in (3) for each x_i. The facet-defining inequalities we identify do not in general map to facet-defining 0–1 inequalities, but they nonetheless yield tighter bounds than known 0–1 cuts.

3 Previous Work

All facets of P_X for a single all-different constraint alldiff(X) are given in [7,17]. If $X = \{x_1,\ldots,x_n\}$ and each x_i has domain $\{v_1,\ldots,v_m\}$ with $n \le m$, they are

$$\sum_{j=1}^{|J|} v_j \le \sum_{i\in J} x_i \le \sum_{j=m-|J|+1}^{m} v_j, \;\; \text{all nonempty } J \subseteq \{1,\ldots,n\} \tag{4}$$

where again $v_1 < \cdots < v_m$. If $m = n$, (4) defines the affine hull when $J = \{1,\ldots,n\}$. The facial structure of a system of two all-different constraints is studied in [1,2].

Facets for general all-different systems are derived for combs in [9,10,12] and for odd holes and webs in [11]. In a conference paper [3], we presented the cycle cuts described here and mapped them into the 0–1 space. The present paper extends the computational tests to benchmark instances, introduces additional families of cuts, and studies the properties of the mapping. Aside from [3], the strategy of mapping finite-domain cuts into the 0–1 space has, to our knowledge, not been investigated, and the cuts we describe here for cycles and paths have not been previously identified. We also generalize the web cuts in [11] and introduce z-cuts for webs.

It is natural to ask when all facets of an all-different system are facets of individual constraints in the system. It is shown in [12] that this occurs if and only if the all-different system has an *inclusion* property, which means that pairwise intersections of sets V_k in the alldiff constraints are ordered by inclusion. The structures studied here lack the inclusion property and therefore generate new classes of facets.

Known facets for the 0–1 graph coloring model are discussed in [4,13,14,16]. These include cuts based on odd holes, webs, anti-webs, cliques, and paths in the given graph G.

Finite-domain cuts have been developed for a few global constraints other than alldiff systems. These include the element constraint [7], the circuit constraint [5], the cardinality constraint [8], cardinality rules [18], the sum constraint [19], and disjunctive and cumulative constraints [8].

4 Cycles

We first investigate valid inequalities that correspond to odd cycles. We define a cycle in graph G to be a subgraph of G induced by the vertices in $V_1, \ldots, V_q \subseteq V$ (for $q \geq 3$), where the subgraph induced by each V_k is a clique, and the only overlapping V_k's are adjacent ones in the cycle $V_1, \ldots, V_q, V_1$. Thus, (for $k < \ell$)

$$V_k \cap V_\ell = \begin{cases} S_k & \text{if } k+1 = \ell \text{ or } (k,\ell) = (1,q) \\ \emptyset & \text{otherwise} \end{cases}$$

where $S_k \neq \emptyset$. A feasible vertex coloring on G must therefore satisfy

$$\text{alldiff}(X_k), \quad k = 1, \ldots, q \tag{5}$$

where again $X_k = \{x_i \mid i \in V_k\}$. The cycle is *odd* if q is odd. If $|V_k| = 2$ for each k, an odd cycle is an *odd hole*.

Figure 1 illustrates an odd cycle with $q = 5$. Each solid oval corresponds to a constraint alldiff(X_k). Thus $V_1 = \{0, 1, 2, 3, 10, 11\}$, and similarly for $V_2, \ldots, V_5$. All the vertices in a given V_k are connected by edges in G.

4.1 Valid Inequalities

We first identify valid inequalities that correspond to a given cycle. In the next section, we show that they are facet-defining.

Lemma 1 *Let $V_1, \ldots, V_q$ induce a cycle, and let $\bar{S}_k \subseteq S_k$ and $|\bar{S}_k| = s \geq 1$ for $k = 1, \ldots, q$. If q is odd and $\bar{S} = \bar{S}_1 \cup \cdots \cup \bar{S}_q$, the following inequality is valid for (1):*

$$\sum_{i \in \bar{S}} x_i \geq \beta(q, s) \tag{6}$$

where

$$\beta(q,s) = \frac{q-1}{2} \sum_{j=0}^{L-2} v_j + \left(sq - \frac{q-1}{2}(L-1) \right) v_{L-1}$$

and

$$L = \left\lceil \frac{sq}{(q-1)/2} \right\rceil$$

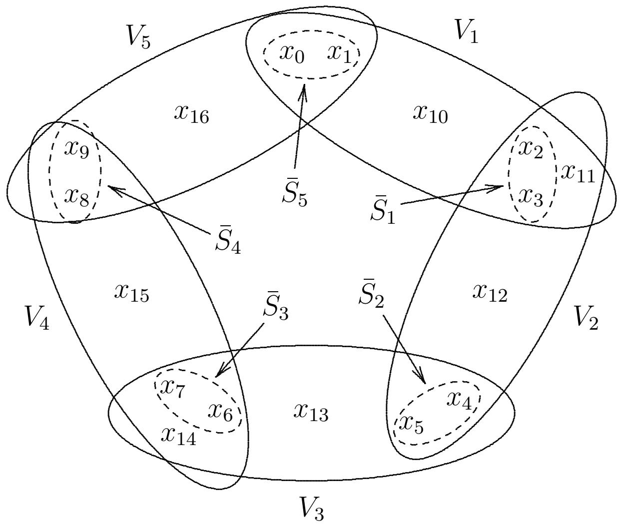

Fig. 1 A 5-cycle. The solid ovals correspond to constraints alldiff(X_k) for $k = 1, \ldots, 5$. The sets $\bar{S}_1, \ldots, \bar{S}_2$ provide the basis for one possible valid cut with $s = 2$.

Proof Because q is odd, each color can be assigned to at most $(q-1)/2$ vertices in the cycle. This means that the vertices must receive at least L distinct colors, and the variables in (5) must take at least L different values. Because $v_0 < \cdots < v_{n-1}$, we have

$$\sum_{i \in \bar{S}} x_i \geq \frac{q-1}{2}(v_0 + v_1 + \cdots + v_{L-2}) + \left(sq - \frac{q-1}{2}(L-1) \right) v_{L-1} = \beta(q,s)$$

where the coefficient of v_{L-1} is the number of vertices remaining to receive color v_{L-1} after colors $v_0, \ldots, v_{L-2}$ are assigned to $(q-1)/2$ vertices each. □

If the cycle is an odd hole, each $|S_k| = 1$ and $L = 3$. So (6) becomes

$$\sum_{i \in \bar{S}} x_i \geq \frac{q-1}{2}(v_0 + v_1) + v_2 \tag{7}$$

If the domain $\{v_0, \ldots, v_{n-1}\}$ of each x_i is $D_\delta = \{0, \delta, 2\delta, \ldots, (n-1)\delta\}$ for some $\delta > 0$, inequality (6) becomes

$$\sum_{i \in \bar{S}} x_i \geq \left(sq - \frac{q-1}{4} L \right) (L-1)\delta \tag{8}$$

for a general cycle and

$$\sum_{i \in \bar{S}} x_i \geq \frac{q+3}{2}\delta$$

for an odd hole.

An example with $q = 5$ appears in Fig. 1. By setting $s = 2$ we can obtain 9 valid inequalities by selecting 2-element subsets $\bar{S}_1$ and $\bar{S}_3$ of S_1 and S_3, respectively.

Here $L = 5$, and if the colors are $0, \ldots, 16$, the right-hand side of the cut is $\beta(5, 2) = 20$. The sets $\bar{S}_1, \ldots, \bar{S}_5$ illustrated in the figure give rise to the valid inequality

$$x_0 + \cdots + x_9 \geq 20 \tag{9}$$

We note that, when q is even, the minimum value of $\sum_{i \in \bar{S}} x_i$ is $\frac{1}{2} q \sum_{i=0}^{2s-1} v_i$. This bound is implied, however, by the sum of all-different inequalities we know from [7] to be facet-defining:

$$\sum_{i \in \bar{S}_k \cup \bar{S}_{k+1}} x_i \geq \sum_{i=0}^{2s-1} v_i, \text{ for } k = 1, 3, \ldots, q-1$$

Thus a valid inequality can be formulated when q is even, but it is not facet-defining.

4.2 Facet-defining Inequalities

We now show that the valid inequalities identified in Lemma 1 are facet-defining. Let the variables x_i for $i \in \bar{S}$ be indexed $x_0, \ldots, x_{qs-1}$. We will say that a partial solution

$$(x_0, x_1, \ldots, x_{qs-1}) = (\bar{x}_0, \bar{x}_1 \ldots, \bar{x}_{qs-1}) \tag{10}$$

is feasible for (1) if it can be extended to a feasible solution of (1). That is, there is a complete solution $(x_0, \ldots, x_{n-1})$ that is feasible in (1) and that satisfies (10). Because $|V|$ colors are available, any partial solution (10) that satisfies (5) can be extended to a feasible solution simply by assigning the remaining vertices distinct unused colors. That is, assign vertices in $V \setminus \{0, \ldots, sq-1\}$ distinct colors from the set $\{v_0, \ldots, v_{n-1}\} \setminus \{\bar{x}_0, \ldots, \bar{x}_{sq-1}\}$.

Theorem 1 *If the graph coloring problem (1) is defined on a graph in which vertex sets $V_1, \ldots, V_q$ induce a cycle, where q is odd, then inequality (6) is facet defining for (1).*

Proof Define

$$F = \{x \text{ feasible for (1)} \mid x \text{ satisfies (6) at equality}\}$$

It suffices to show that if $\mu x = \mu_n$ holds for all $x \in F$, then there is a scalar $\lambda > 0$ such that

$$\mu_i = \begin{cases} \lambda & \text{for } i = 0, \ldots, qs-1 \\ \beta(q,s)\lambda & \text{for } i = n \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

We will construct a partial solution $(\bar{x}_0, \ldots, \bar{x}_{qs-1})$ that is feasible for (1) as follows. Domain values $v_0, \ldots, v_{L-2}$ will occur $(q-1)/2$ times in the solution, and domain value v_{L-1} will occur r times, where

$$r = qs - \frac{q-1}{2}(L-1)$$

This will ensure that (6) is satisfied at equality. We form the partial solution by first cycling r times through the values $v_0, \ldots, v_{L-1}$, and then by cycling through the values $v_0, \ldots, v_{L-2}$. Thus

$$\bar{x}_i = \begin{cases} v_{i \bmod L} & \text{for } i = 0, \ldots, rL-1 \\ v_{(i-rL) \bmod (L-1)} & \text{for } i = rL, \ldots, qs-1 \end{cases} \quad (12)$$

To show that this partial solution is feasible for the odd cycle, we must show

$$\begin{array}{ll} \text{alldiff}\{\bar{x}_i,\ i \in \bar{S}_k \cup \bar{S}_{k+1}\}, \text{ for } k = 1, \ldots, q-1 & (a) \\ \text{alldiff}\{\bar{x}_i,\ i \in \bar{S}_1 \cup \bar{S}_q\} & (b) \end{array}$$

To show (a), we note that the definition of L implies $L - 1 \geq 2s$. Therefore, any sequence of $2s$ consecutive $\bar{x}_i$'s are distinct, and (a) is satisfied. To show (b), we note that the number of values $\bar{x}_{rL}, \ldots, \bar{x}_{qs-1}$ is

$$(qs-1) - rL + 1 = (L-1)\left(\frac{q-1}{2}L - qs\right)$$

from the definition of r. Because the number of values is a multiple of $L-1$, the values $\bar{x}_i$ for $i \in \bar{S}_q$ are $(\bar{x}_{(q-1)s}, \ldots, \bar{x}_{qs-1}) = (v_{L-s-1}, \ldots, v_{L-2})$, and they are all distinct. The values $\bar{x}_i$ for $i \in \bar{S}_1$ are $(\bar{x}_0, \ldots, \bar{x}_{s-1}) = (v_0, \ldots, v_{s-1})$ and are all distinct. But $L - 1 \geq 2s$ implies $L - s > s$, and (b) follows.

We now construct a partial solution $(\tilde{x}_0, \ldots, \tilde{x}_{qs-1})$ from the partial solution in (12) by swapping any two values $\bar{x}_\ell, \bar{x}_{\ell'}$ for $\ell, \ell' \in \bar{S}_k \cup \bar{S}_{k+1}$, for any $k \in \{1, \ldots, q-1\}$. That is,

$$\tilde{x}_i = \begin{cases} \bar{x}_{\ell'} & \text{if } i = \ell \\ \bar{x}_\ell & \text{if } i = \ell' \\ \bar{x}_i & \text{otherwise} \end{cases} \quad (13)$$

Extend the partial solutions (12) and (13) to complete solutions $\bar{x}$ and $\tilde{x}$, respectively, by assigning values with

$$\bar{x}_i = \tilde{x}_i \text{ for } i \notin \{0, \ldots, qs-1\}$$

such that the values assigned to $\bar{x}_i$ for $i \notin \{0, \ldots, qs-1\}$ are all distinct and do not belong to $\{v_0, \ldots, v_{L-1}\}$. Because $\bar{x}$ and $\tilde{x}$ are feasible and satisfy (6) at equality, they satisfy $\mu x = \mu_{n+1}$. So we have $\mu\bar{x} = \mu\tilde{x}$, which implies $\mu_\ell = \mu_{\ell'}$ for $\ell, \ell' \in \bar{S}_k \cup \bar{S}_{k+1}$ for any pair $\ell, \ell' \in \bar{S}_k \cup \bar{S}_{k+1}$ and any $k \in \{1, \ldots, q-1\}$. This implies

$$\mu_\ell = \mu_{\ell'} \text{ for any } \ell, \ell' \in \bar{S} \quad (14)$$

Define $\bar{x}'$ by letting $\bar{x}' = \bar{x}$ except that for an arbitrary $\ell \notin \{0, \ldots, qs-1\}$, $\bar{x}'_\ell$ is assigned a value that does not appear in the tuple $\bar{x}$. Since $\bar{x}$ and $\bar{x}'$ are feasible and satisfy (6) at equality, we have $\mu\bar{x} = \mu\bar{x}'$. This and $\bar{x}_\ell \neq \bar{x}'_\ell$ imply

$$\mu_i = 0, \ \ i \in V \setminus \{0, \ldots, qs-1\} \quad (15)$$

Finally, (14) implies that for some $\lambda > 0$,

$$\mu_i = \lambda, \ \ i = 0, \ldots, qs-1 \quad (16)$$

Because $\mu\bar{x} = \mu_n$, we have from (16) that $\mu_n = \beta(q, s)\lambda$. This, (15), and (16) imply (11). □

In the example of Fig. 1, suppose that the vertices in $V_1, \ldots, V_5$ induce a cycle of G. That is, all vertices in each V_k are connected by edges, and there are no other edges of G between vertices in $V_1 \cup \cdots \cup V_5$. Then (9) is facet-defining for (1).

4.3 Bounds on the Chromatic Number

We can write a valid inequality involving the objective function variable z if the domain of each x_i is D_δ for $\delta > 0$. To do so we rely on the following, where $e = (1, \ldots, 1)$:

Theorem 2 *Suppose $ax \geq \beta$ is valid for a graph coloring problem (1) in which each x_i has domain D_δ for $\delta > 0$. Then if $ae \geq 0$, the inequality*

$$aez \geq ax + \beta \tag{17}$$

is also valid for (1).

Proof To show that (17) is valid, note that for any $x \in D_\delta^n$, $z - x_i \in D_\delta$ for all i, where $z = \max_i\{x_i\}$. Because $ax \geq \beta$ is valid for all $x \in D_\delta^n$ and $z - x_i \in D_\delta$, $ax \geq \beta$ holds when $z - x_i$ is substituted for each x_i. Thus (17) holds when $z = \max_i\{x_i\}$. But because z in (1) must satisfy $z \geq x_i$ for each i, and $ae \geq 0$, it follows that (17) holds for (1). □

Inequality (6) and Theorem 2 imply

Corollary 1 *If the graph coloring problem (1) is defined on a graph in which vertex sets $V_1, \ldots, V_q$ induce a cycle, where q is odd and each x_i has domain D_δ with $\delta > 0$, then*

$$z \geq \frac{1}{qs} \sum_{i \in \bar{S}} x_i + \frac{\beta(q,s)}{qs} \tag{18}$$

is valid for (1), where

$$\frac{\beta(q,s)}{qs} = \left(1 - \frac{q-1}{4qs} L\right)(L-1)\delta$$

In the case of an odd hole ($s = 1$), the z-cut is

$$z \geq \frac{1}{q} \sum_{i \in \bar{S}} x_i + \frac{q+3}{2q}\delta$$

In the example of Fig. 1, the z-cut is

$$z \geq \tfrac{1}{10}\,(x_0 + \cdots + x_9) + 2 \tag{19}$$

It is straightforward to deduce the linear programming (LP) bound on the chromatic number that is provided by x-cuts and/or z-cuts in the finite-domain space, when no other cuts are present. The bound is the minimum of $z + 1$ subject to the cut(s) and $z \geq x_i$ for all $i \in V$.

Lemma 2 *Let the x-cut $ax \geq \beta$ and the z-cut $z \geq ax/ae + \beta/ae$ be valid for an arbitrary coloring problem (1) with domain D_1, where $ae > 0$. The x-cut and z-cut yield the lower bound $\beta/ae + 1$ on the chromatic number when used individually, and the lower bound $2\beta/ae + 1$ when used together.*

Proof The optimal value of the LP that contains the x-cut is $z + 1 = \beta/ae + 1$, because this and $x_i = \beta/ae$ for all i is a primal feasible solution, and an assignment of multiplier $1/ae$ to the x-cut and a_i/ae to each constraint $z \geq x_i$ is dual feasible. The optimal value of the LP with the z-cut is likewise $z + 1 = \beta/ae + 1$, because this and $x_i = 0$ for all i is primal feasible, and an assignment of multiplier 1 to the z-cut and zero to each constraint $z \geq x_i$ is dual feasible. The optimal value with both cuts is $z + 1 = 2\beta/ae + 1$, because this and $x_i = \beta/ae$ is primal feasible, and an assignment of multiplier $1/ae$ to the x-cut, 1 to the z-cut, and zero to each $z \geq x_i$ is dual feasible. □

For domain D_1, the x-cut and z-cut for a cycle provide the bound $\beta(q, s)/qs+1$ when used individually, and the bound $2\beta(q, s)/qs + 1$ when used together.

4.4 Mapping to 0–1 Cuts

The 0–1 model for a coloring problem on a cycle has the following continuous relaxation:

$$\begin{aligned}
&\sum_{j\in J} y_{ij} = 1, \;\; i = 0, \ldots, n-1 && (a)\\
&\sum_{i\in V_k} y_{ij} \leq w_j, \;\; j \in J, \; k = 1, \ldots, q && (b)\\
&0 \leq y_{ij}, w_j \leq 1, \;\; \text{all } i, j && (c)
\end{aligned} \tag{20}$$

Because constraints (b) appear for each maximal clique, the relaxation implies all clique inequalities $\sum_{i\in V_k} y_{ij} \leq 1$. Nonetheless, we will see that two finite-domain cuts strengthen the relaxation more than the collection of all odd-hole cuts.

To simplify the discussion, let each x_i have domain $D_1 = \{0, 1, \ldots, n-1\}$. As noted earlier, we can map an x-cut $ax \geq \beta$ into 0–1 space by substituting $\sum_j jy_{ij}$ for each x_i, which yields

$$\sum_{ij} ja_i y_{ij} \geq \beta \tag{21}$$

Because the minimum color number is zero, minimizing the number of colors $\sum_j w_j$ is the same as minimizing $z + 1$, where z is the largest color number. The z-cut $z \geq ax/ae + \beta/ae$ therefore maps into

$$\sum_j w_j \geq \frac{1}{ae}\sum_{ij} ja_i y_{ij} + \frac{\beta}{ae} + 1 \tag{22}$$

We first note that the x-cut and z-cut, used together, yield an LP bound in 0–1 space at least as tight as in finite-domain space. The LP bound in 0–1 space is the minimum of $\sum_j w_j$ subject to (20)–(22).

Lemma 3 *Suppose that x-cut $ax \geq \beta$ and z-cut $z \geq (ax + \beta)/ae$ are valid for an arbitrary coloring problem (1). If they are mapped into 0–1 space, they yield an LP bound on the chromatic number of at least $2\beta/ae + 1$ when used together.*

Proof This is seen by taking a linear combination of (21) and (22), with weight $1/ae$ on the former and 1 on the latter. □

The x-cut (8) maps into the cut

$$\sum_{i\in\bar{S}}\sum_{j=1}^{n-1} jy_{ij} \geq \left(sq - \frac{q-1}{4}L\right)(L-1). \quad (23)$$

The z-cut (18) maps into

$$\sum_{j=0}^{n-1} w_j - 1 \geq \frac{1}{qs}\sum_{i\in\bar{S}}\sum_{j=1}^{n-1} jy_{ij} + \frac{\beta(q,s)}{qs}. \quad (24)$$

We will compare cuts (23)–(24) with classical odd-hole cuts, which have the form

$$\sum_{i\in H} y_{ij} \leq \frac{q-1}{2}w_j, \;\; j = 0,\ldots,n-1 \quad (25)$$

where H is the vertex set for an odd hole. The cut (25) is not facet defining in general, although it is facet defining when H contains all vertices of G ([15], page 261). This is in contrast with the finite-domain cut (6), which is facet defining in the x-space for any odd hole in G (and more generally, any odd cycle in G).

We first note that when $s = 1$, the 0–1 x-cut (23) is redundant of odd-hole cuts.

Lemma 4 *If $s = 1$, the 0–1 x-cut (23) is implied by the 0–1 model (20) with odd-hole cuts (25).*

Proof When $s = 1$, the cut (23) becomes

$$\sum_{i\in\bar{S}}\sum_{j=0}^{n-1} jy_{ij} \geq \frac{q+3}{2} \quad (26)$$

It suffices to show that (26) is dominated by a nonnegative linear combination of (20) and (25), where $H = \bar{S}$ in (25). Assign multiplier 2 to each constraint in (20a); multipliers 2 and 1, respectively, to constraints (25) with $j = 0, 1$; and multipliers $q-1$ and $(q-1)/2$, respectively, to the constraints $w_0 \leq 1$ and $w_1 \leq 1$. The resulting linear combination is

$$\sum_{i\in\bar{S}} y_{i1} + 2\sum_{j=2}^{n-1}\sum_{i\in\bar{S}} y_{ij} \geq 2q - \frac{q-1}{2} - (q-1) = \frac{q+3}{2}$$

This dominates (26) because the left-hand side coefficients are less than or equal to the corresponding coefficients in (26). □

However, the two finite-domain cuts (23) and (24), when combined, provide a tighter bound than the n odd-hole cuts (25) even when $s = 1$. We first establish the strength of odd-hole cuts.

Lemma 5 *When $s = 1$, odd-hole cuts associated with an odd q-cycle yield the LP bound $2q/(q-1)$ when added to the 0-1 model (20). When $s > 1$, they have no effect on the LP bound $2s$ that follows from (20).*

Proof First suppose $s = 1$. A linear combination of the q constraints (20a) with weight $-2/(q-1)$ and the odd-hole cuts (25) with weight $2/(q-1)$ yields the bound $\sum_j w_j \geq 2q/(q-1)$. Because $n = sq = q$, the same value is obtained from the primal feasible solution $y_{ij} = 1/q$ and $w_j = 2/(q-1)$ and is therefore optimal in the LP. Now suppose $s > 1$. A linear combination of the $n = qs$ constraints (20a) with weight $-1/q$ and the qn constraints (20b) with weight $1/2q$ yields the bound $\sum_j w_i \geq 2s$. The primal solution $y_{ij} = 1/n$ and $w_j = 2s/n$ yields the same value and is feasible in (20). It also satisfies each odd-hole cut (25) because the left-hand side is q/n and the right-hand side is $s(q-1)/n$. But $s \geq 2$ and $q \geq 3$ imply that $q/n \leq s(q-1)/n$. Thus $2s$ is the LP bound even when odd-hole cuts are added. □

Corollary 2 *The two finite-domain cuts (23) and (24) yield a strictly tighter LP bound (except when $q = 3, s = 1$, whereupon the bound is identical) on the chromatic number for an odd cycle than all clique cuts and all ns^q odd-hole cuts for the cycle.*

Proof Lemma 3 implies that the two finite-domain cuts yield a bound equal to $2\beta(q,s)/qs+1$. When $s = 1$, this bound is $(2q+3)/q$, which is greater than $2q/(q-1)$ when $q > 3$ and equivalent when $q = 3$. When $s > 1$, we observe that since $L = \lceil 2qs/(q-1) \rceil$, can write $L = 2qs/(q-1) + \Delta$ where $0 \leq \Delta < 1$. Thus we have

$$\begin{aligned}\frac{2\beta(q,s)}{qs} + 1 &= \frac{2}{qs}\left(qs - \frac{q-1}{4}L\right)(L-1) + 1 \\ &= \frac{2}{qs}\left(qs - \frac{q-1}{4}\left(\frac{2qs}{q-1} + \Delta\right)\right)\left(\frac{2qs}{q-1} + \Delta - 1\right) + 1 \\ &= \frac{2qs}{q-1} + \frac{q-1}{2qs}\Delta(1-\Delta) \geq \frac{2qs}{q-1} > 2s\end{aligned}$$

where the first equality is from the definition of $\beta(q,s)$ and the first inequality from $0 \leq \Delta < 1$. Thus the finite-domain bound is strictly greater than the odd-hole bound for all s. Finally, There are s^q odd-hole cuts for each color j, one for every H that selects one element from each S_k, $k = 1, \ldots, q$. □

For example, when $q = 5$ and $s = 2$, there are $ns^q = 320$ odd-hole cuts. The lower bound on the chromatic number is $2s = 4.0$ with or without them. However, the two finite-domain cuts (23) and (24) yield a bound of $2\beta(q,s)/qs+1 = 5.0$. This bound is actually sharp in the present instance, because the chromatic number is 5. Thus two finite-domain cuts significantly improve the bound, while 320 odd-hole cuts have no effect on the bound. Further comparisons appear in Section 7.

Odd-hole cuts for cycles can be lifted to obtain stronger cuts by exploiting an observation from the proof of Lemma 1. The proof uses the fact that each color can be assigned to at most $(q-1)/2$ vertices in the cycle. This implies the valid cuts

$$\sum_{i \in V_1 \cup \cdots \cup V_q} y_{ij} \leq \frac{q-1}{2} w_j, \text{ all } j \in J \tag{27}$$

This observation can be made, and the lifted cuts obtained, without reference to finite-domain cuts. However, any valid bound obtain by finite-domain analysis can in principle be be obtained by other means, for example by selecting a suitable set of facet-defining inequalities in 0–1 space. The difficulty lies in identifying the inequalities. The advantage of finite-domain analysis is that it may discover valid cuts that were previously unknown. To our knowledge, no one has observed that the finite-domain cuts (23)–(24) or the lifted odd-hole cuts (27) are valid for graph coloring.

Furthermore, two finite-domain cuts provide generally tighter bounds in less time than the lifted odd-hole cuts. It is easy to show that the lifted cuts (27) yield a bound of $2qs/(q-1)$, which has the following relationship to the finite-domain bound:

Lemma 6 *The bound* $2qs/(q-1)$ *is strictly weaker than the finite domain bound of* $2\beta(q,s)/qs+1$ *except when* s *is an integer multiple of* $(q-1)/2$*, in which case the two bounds are equal.*

Proof When s is an integer multiple of $(q-1)/2$, we have $L = 2qs/(q-1)$, and direct substitution verifies that the two bounds are equal. Otherwise, we have $L = 2qs/(q-1) + \Delta$ for $0 < \Delta < 1$. Substitution of this into the formula for $\beta(q,s)$ yields the finite-domain bound

$$\frac{2qs}{q-1} + \frac{q-1}{2qs}\Delta(1-\Delta) > \frac{2qs}{q-1}$$

and the lemma follows. □

The difference between the two bounds is never great, but the finite domain bound is achieved with only 2 inequalities, as opposed to n lifted inequalities (27). We will see in Section 7 that this allows the finite-domain bound to be obtained in substantially less computation time.

5 Webs

We next study cuts that arise from webs. A web $W(q,r)$ is a graph in which vertices can be arranged cyclically so that the edges connect pairs of vertices separated by a distance of at least r on the cycle. More formally, given that $q \geq 2r+1$ and $r \geq 1$, a web $W(q,r)$ is a graph on vertices $0,\ldots,q-1$ whose edges are all (i,i') such that $0 \leq i \leq q-r-1$ and $r \leq i'-i \leq q-r$. Thus $W(q,1)$ is a clique. When q is odd, $W(q,\frac{q-1}{2})$ is an odd hole, and $W(q,2)$ is an odd anti-hole (the complement of an odd hole). Figure 2 illustrates $W(7,2)$.

We will focus on webs for which q and d are coprime, where $d = q \bmod r$ is the remainder after dividing q by r. This implies, in particular, that q and r are coprime. Odd holes and odd anti-holes are special cases of these webs. Such webs give rise to 0–1 finite-domain cuts that provide tighter bounds than standard web cuts.

5.1 Facet-Defining Inequalities

Theorem 3 *Let vertices* $0,\ldots,q-1$ *of* G *induce a web* $W(q,r)$ *, where* q *and* $d = q \bmod r$ *are coprime. The following inequality is facet-defining for (1):*

$$\sum_{i=0}^{q-1} x_i \geq \gamma(q,r) \tag{28}$$

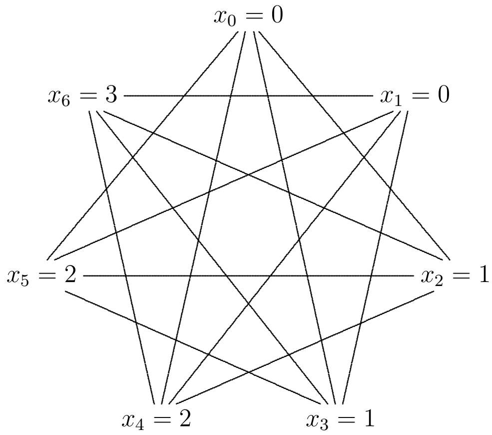

Fig. 2 Web $W(7,2)$, which is an odd antihole. Variables connected by an edge appear in a common alldiff constraint. A feasible solution is shown.

where

$$\gamma(q,r) = r\sum_{j=0}^{t-1} v_j + (q-tr)v_t$$

and $t = \lfloor q/r \rfloor$.

Proof To show that (28) is valid, we observe that each color can be used at most r times. This is because if any set of r vertices receive color j, no two of these vertices can be separated by distance of r or more in the cycle, because any such pair of vertices are connected. The vertices must therefore be adjacent. Because every other vertex is connected to one of them, no other vertex can receive color j, and no more than r vertices can receive color j. This means that at least $t+1$ colors must be used. Thus

$$\sum_{i=0}^{q-1} x_i \geq r(v_0 + v_1 + \cdots + v_{t-1}) + dv_t = \gamma(q,r)$$

where $d = q - tr$ is the number of vertices remaining after assigning each of the colors $v_0, \ldots, v_{t-1}$ to r vertices.

To show that (28) is facet defining, let

$$F = \{x \text{ feasible for (1)} \mid x \text{ satisfies (28) at equality}\}$$

It suffices to show that if $\mu x = \mu_n$ holds for all $x \in F$, then there is a scalar $\lambda > 0$ such that

$$\mu_i = \begin{cases} \lambda & \text{for } i = 0, \ldots, q-1 \\ \gamma(q,r)\lambda & \text{for } i = n \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

The partial solution

$$(\bar{x}_0, \ldots, \bar{x}_{q-1}) = \Big(\underbrace{v_0, \ldots, v_0}_{r}, \underbrace{v_1, \ldots, v_1}_{r}, \ldots, \underbrace{v_{t-1}, \ldots, v_{t-1}}_{r}, \underbrace{v_t, \ldots, v_t}_{d}\Big) \tag{30}$$

is clearly feasible and satisfies (28) at equality. We construct a partial solution $(\tilde{x}_0, \ldots, \tilde{x}_{q-1})$ from (30) by swapping the color assignment of the vertices receiving color v_t with that of the last d vertices receiving color v_0. That is, we let

$$\tilde{x}_i = \begin{cases} v_t & \text{if } i \in \{r-d, \ldots, r-1\} \\ v_0 & \text{if } i \in \{q-d, \ldots, q-1\} \\ \bar{x}_i & \text{otherwise} \end{cases}$$

Note that $(\tilde{x}_0, \ldots, \tilde{x}_{q-1})$ is feasible, because colors $v_1, \ldots, v_{t-1}$ are assigned to r adjacent vertices as before, color v_0 is assigned to the r adjacent vertices $q - d, \ldots, q-1, 0, \ldots, r-d-1$, and color v_t is assigned to the remaining adjacent vertices $r-d, \ldots, r-1$. Extend the two partial solutions to feasible solutions $\bar{x}$ and $\tilde{x}$ of (1). Because $\bar{x}$ and $\tilde{x}$ satisfy (28) at equality, we have $\mu\bar{x} = \mu\tilde{x}$, which yields

$$\mu_{r-d} + \cdots + \mu_{r-1} = \mu_{q-d} + \cdots + \mu_{q-1}$$

By symmetry, we conclude

$$\mu_i + \cdots + \mu_{(i+d-1) \bmod q} = \mu_{(i-r) \bmod q} + \cdots + \mu_{(i-r+d-1) \bmod q}$$

for $i = 0, \ldots, q-1$. Because q and r are coprime, this implies that the sums $\mu_i + \cdots + \mu_{(i+d-1) \bmod q}$ are equal for all i. Thus, in particular, they are equal for i and $i+1$, which yields $\mu_i = \mu_{(i+d) \bmod q}$ for all i. Because q and d are coprime, this implies that $\mu_0 = \cdots = \mu_{q-1}$ and

$$\mu_i = \lambda, \;\; i = 0, \ldots, q-1 \tag{31}$$

for some $\lambda > 0$.

Define $\bar{x}'$ by letting $\bar{x}' = \bar{x}$ except that for an arbitrary $\ell \notin \{0, \ldots, q-1\}$, $\bar{x}'_\ell$ is assigned a value that does not appear in the tuple $\bar{x}$. Since $\bar{x}, \bar{x}' \in F$, we have $\mu\bar{x} = \mu\bar{x}'$, which implies

$$\mu_i = 0, \; i \in V \setminus \{0, \ldots, q-1\} \tag{32}$$

Because $\mu\bar{x} = \mu_n$, we have from (31) that $\mu_n = \gamma(q, r)\lambda$. Thus, (31) and (32) imply (29). □

For domain D_δ with $\delta > 0$, the cut (28) is

$$\sum_{i=0}^{q-1} x_i \geq \left(q - \tfrac{1}{2}(t+1)r\right) t\delta \tag{33}$$

For an odd antihole $W(q, 2)$ with domain D_δ, the cut simplifies to

$$\sum_{i=0}^{q-1} x_i \geq \tfrac{1}{4}(q-1)^2\delta \tag{34}$$

5.2 Bounds on the Chromatic Number

Theorems 2 and 3 imply

Corollary 3 *If the graph coloring problem (1) is defined on a graph in which the vertices in a subcollection of the vertex sets V_k induce a web $W(q,r)$, where q and r are coprime, and each x_i has domain D_δ with $\delta > 0$, then*

$$z \geq \frac{1}{q}\sum_{i=0}^{q-1} x_i + \left(1 - \frac{r}{2q}(t+1)\right) t\delta \tag{35}$$

is valid for (1), where $t = \lfloor q/r \rfloor$.

Lemma 2 implies that for domain D_1, the x-cut (33) and z-cut (35) yield the LP bound $t-(rt/2q)(t+1)+1$ when used individually, and $2t-(rt/q)(t+1)+1$ when used together. For example, the antihole of Fig. 2 gives rise to the facet-defining cuts

$$\sum_{i=0}^{6} x_i \geq 9, \quad z \geq \tfrac{1}{7}\sum_{i=0}^{6} x_i + \tfrac{9}{7}$$

For a graph consisting of this web, these cuts yield the bound $2\frac{2}{7}$ when used individually and $3\frac{4}{7}$ when used together.

5.3 Mapping to 0–1 Cuts

If each x_i has domain $D_1 = \{0, 1, \ldots, n-1\}$ the x-cut (33) maps into the cut

$$\sum_{i=0}^{q-1}\sum_{j=1}^{n-1} j y_{ij} \geq \left(q - \tfrac{1}{2}(t+1)r\right) t \tag{36}$$

when $\delta = 1$. The z-cut (35) maps into

$$\sum_{j=0}^{n-1} w_j - 1 \geq \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=1}^{n-1} j y_{ij} + \left(1 - \frac{r}{2q}(t+1)\right) t \tag{37}$$

We wish to compare these cuts with known cuts for webs. Facet-defining web cuts for a 0–1 model of the coloring problem are given in [16]. These cuts are defined in a space in which the variables correspond to edges and colorings correspond to "admissible star partitions" of the graph. However, the cuts are based on the fact that at most r vertices can be assigned any given color, and we can write analogous web cuts in the y_{ij}-space:

$$\sum_{i=0}^{q-1} y_{ij} \leq r \cdot w_j, \text{ all } j \tag{38}$$

We first derive the bound obtained from these cuts. Note that $n = q$ when G is a web.

Lemma 7 *The q web cuts (38) yield the LP bound q/r when added to the 0–1 graph coloring model (20) for a web $W(q,r)$.*

Proof A linear combination of the q inequalities (20a) with weight $-1/r$ each, and the q web cuts (38) with weight $1/r$ each, yields the bound $\sum_j w_j \geq q/r$. It suffices to show that the primal solution $y_{ij} = 1/q$ and $w_j = 1/r$ is feasible, because it has value $\sum_j w_j = q/r$. It is clearly feasible in (20a) and the web cuts (38). We must show that it is feasible in (20b) for each maximal clique V_k in the web. But because every maximal clique V_k in $W(q,r)$ has size $\lfloor q/r \rfloor$, the left-hand side of (20b) is $|V_k|(1/q) < 1/r$, and the constraint is satisfied. □

Now we can show that the finite-domain cuts are stronger.

Corollary 4 *The two finite-domain cuts (36) and (37) provide a strictly tighter LP bound on the chromatic number for a web $W(q,r)$ than the n web cuts (38).*

Proof By Lemma 3, the two finite-domain cuts (36)–(37) provide bound

$$2t - \frac{rt}{q}(t+1) + 1 = \frac{q}{r} + \frac{r}{q}\alpha(1-\alpha) > \frac{q}{r}$$

where the equality follows from setting $\alpha = q/r - t$, and the inequality follows from the fact that $0 < \alpha < 1$. Because the web cuts yield the bound q/r by Lemma 7, the corollary follows. □

For example, for an antihole $W(7,2)$, seven 0–1 web cuts (38) give a bound of 3.5, while the two finite-domain cuts provide a bound of 3.5714. Further comparisons appear in Section 7.

6 Paths

Paths present an interesting case because they give rise to finite-domain cuts that are redundant in the 0–1 model. That is, they have no effect on the bound when the problem consists entirely of a path system. However, a few finite-domain cuts may replace a large number of inequalities in a more complex coloring problem, allowing a substantial reduction in the size of the 0–1 model. In addition, path cuts are useful in a finite-domain model of the problem.

A path is a subgraph of G induced by the vertices in subsets $V_1, \ldots, V_q$ of V, provided the subgraph induced by each V_k is a clique, and only adjacent V_k's overlap. That is,

$$V_k \cap V_\ell = \begin{cases} S_k & \text{if } k+1 = \ell \\ \emptyset & \text{otherwise} \end{cases}$$

where $S_k \neq \emptyset$. We let the sets V_k (and S_k) denote both the vertices i in the sets and their corresponding variables x_i.

6.1 Facet-defining Inequalities

Facet-defining inequalities can be obtained by selecting one variable from each overlap S_k. Valid cuts can be obtained if two or more variables are selected, as with cycles, but they are redundant of the single-variable cuts.

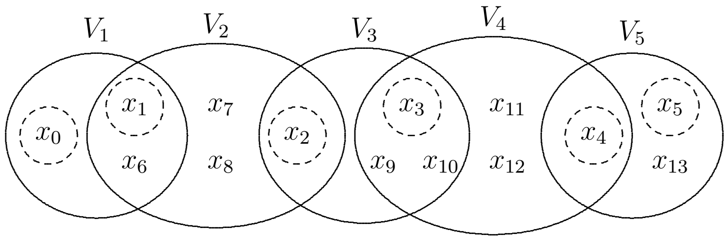

Fig. 3 A path with $q = 5$. The variables in dashed circles appear in one possible valid cut.

Lemma 8 *Let $V_1, \ldots, V_q$ be a path, and let $0 \in V_1 \setminus V_2$, $q \in V_q \setminus V_{q-1}$, and $i \in S_i$ for $i = 1, \ldots, q-1$. If q is odd, the following inequality is valid for (1):*

$$(v_2 - v_0)(x_0 + x_q) + (v_1 - v_0)\sum_{i=1}^{q-1} x_i \geq \phi(q) \tag{39}$$

where

$$\phi(q) = \left(\frac{q-1}{2}(v_1 - v_0) + v_2 - v_0\right)(v_0 + v_1)$$

Proof Suppose to the contrary (39) is not satisfied, which implies

$$(v_1 - v_0)\sum_{i=1}^{q-1} x_i < \phi(q) - (v_2 - v_0)(x_0 + x_q) \tag{40}$$

Adjacent variables in the list $x_0, \ldots, x_q$ must take distinct values. So we have $x_i + x_{i+1} \geq v_0 + v_1$ for $i = 1, 3, \ldots, q-2$. Summing these, we obtain

$$\sum_{i=1}^{q-1} x_i \geq \frac{q-1}{2}(v_0 + v_1)$$

This and (40) imply

$$\frac{q-1}{2}(v_1 - v_0)(v_0 + v_1) < \left(\frac{q-1}{2}(v_1 - v_0) + v_2 - v_0\right)(v_0 + v_1) - (v_2 - v_0)(x_0 + x_q)$$

which implies $x_0 + x_q < v_0 + v_1$. This is possible only if $x_0 = x_q = v_0$, because $v_1 > v_0 \geq 0$. Thus $x_1, x_{q-1} \neq v_0$, which means that at most $(q-3)/2$ of the variables $x_1, \ldots, x_{q-1}$ can take the value v_0, and at least one variable must take a value larger than v_1. Thus we have

$$(v_1 - v_0)\sum_{i=1}^{q-1} x_i \geq (v_1 - v_0)\left(\frac{q-3}{2}v_0 + \frac{q-1}{2}v_1 + v_2\right) = \phi(q) - 2v_0(v_2 - v_0)$$

But this is inconsistent with (40) because $v_2 - v_0 \geq 0$ and $x_0 + x_q \geq 2v_0$. □

If each x_i has domain D_δ for $\delta > 0$, the cut (39) is

$$2(x_0 + x_q) + \sum_{i=1}^{q-1} x_i \geq \tfrac{1}{2}(q+3)\delta \tag{41}$$

We can now show the following (see also Theorem 6 of [11]).

Theorem 4 *If the graph coloring problem (1) is defined on a graph in which vertex sets $V_0, \ldots, V_q$ induce a path, where q is odd, then inequality (39) is facet defining for (1).*

Proof Let

$$F = \{x \text{ feasible for (1)} \mid x \text{ satisfies (39) at equality}\}$$

It suffices to show that if $\mu x \geq \mu_{n+1}$ holds for all $x \in F$, there is a scalar $\lambda > 0$ such that

$$\mu_i = \begin{cases} (v_1 - v_0)\lambda & \text{for } i = 1, \ldots, q-1 \\ (v_2 - v_0)\lambda & \text{for } i = 0, q \\ \phi(q)\lambda & \text{for } i = n+1 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

The following partial solutions are feasible for (1):

$$\begin{aligned} (\bar{x}_0, \ldots, \bar{x}_q) &= (v_1, v_0, v_1, v_0, v_1, v_0, v_1, \ldots, v_1, v_0) \\ (\hat{x}_0, \ldots, \hat{x}_q) &= (v_0, v_2, v_1, v_0, v_1, v_0, v_1, \ldots, v_1, v_0) \\ (\tilde{x}_0, \ldots, \tilde{x}_q) &= (v_0, v_1, v_2, v_0, v_1, v_0, v_1, \ldots, v_1, v_0) \end{aligned}$$

They can be extended to complete solutions $\bar{x}, \hat{x}, \tilde{x}$ with

$$\bar{x}_i = \hat{x}_i = \tilde{x}_i \text{ for } i \notin \{0, \ldots, q\}$$

in the manner described above. Because $\hat{x}$ and $\tilde{x}$ satisfy (39) at equality, they satisfy $\mu x = \mu_{n+1}$, and we have $\mu\hat{x} = \mu\tilde{x} = \mu_{n+1}$. This implies $\mu(\hat{x} - \tilde{x}) = 0$, and therefore $\mu_1 = \mu_2$.

Note that we obtained $\tilde{x}$ from $\hat{x}$ by switching the values of x_1 and x_2. We can similarly show that $\mu_2 = \mu_3$ by switching the values of x_2 and x_3, and so forth, yielding

$$\mu_1 = \cdots = \mu_{q-1} \tag{43}$$

Also $\bar{x}$ satisfies (39) at equality, and we have $\mu\bar{x} = \mu\hat{x} = \mu_{n+1}$. This implies $(v_1 - v_0)\mu_0 = (v_2 - v_0)\mu_1$. If we define $\bar{x}$ and $\hat{x}$ by

$$\begin{aligned} (\bar{x}_0, \ldots, \bar{x}_q) &= (v_0, v_1, v_0, v_1, \ldots, v_0, v_1, v_0, v_1) \\ (\hat{x}_0, \ldots, \hat{x}_q) &= (v_0, v_1, v_0, v_1, \ldots, v_0, v_1, v_2, v_0) \end{aligned}$$

then $\bar{x}$ still satisfies (39) at equality, and we get $(v_1 - v_0)\mu_q = (v_2 - v_0)\mu_{q-1}$. Thus by (43)

$$(v_1 - v_0)\mu_0 = (v_1 - v_0)\mu_q = (v_2 - v_0)\mu_i, \ i = 1, \ldots, q-1 \tag{44}$$

Now define $\bar{x}'$ by letting $\bar{x}' = \bar{x}$ except that for an arbitrary $\ell \notin \{0, \ldots, q\}$, $\bar{x}'_\ell$ is assigned a value that does not appear in the tuple $\bar{x}$. Since $\bar{x}$, $\bar{x}'$ are feasible and satisfy (39) at equality, we have $\mu\bar{x} = \mu\bar{x}'$. This and $\bar{x}_\ell \neq \bar{x}'_\ell$ imply

$$\mu_i = 0, \ i \in V \setminus \{0, \ldots, q\} \tag{45}$$

Finally, (44) implies that for some $\lambda > 0$,

$$\mu_0 = \mu_q = (v_2 - v_0)\lambda \text{ and } \mu_i = (v_1 - v_0)\lambda, \ \ i = 1, \ldots, q-1 \tag{46}$$

Because $\mu\bar{x} = \mu_{n+1}$, we have from (46) that $\mu_{n+1} = \phi(q)\lambda$. This, (45), and (46) imply (42). $\square$

6.2 Bounds on the Chromatic Number

Theorems 2 and 4 imply

Corollary 5 *If the graph coloring problem (1) is defined on a graph in which vertex sets $V_0, \ldots, V_q$ induce a path, where q is odd and each x_i has domain D_δ for $\delta > 0$, then*

$$z \geq \frac{1}{q+3}\left(2(x_0 + x_q) + \sum_{i=1}^{q-1} x_i\right) + \frac{\delta}{2} \tag{47}$$

is valid for (1).

If the colors are $0, 1, \ldots, 5$, the cuts for the path in Fig. 3 are

$$2(x_0 + x_5) + \sum_{i=1}^{4} x_i \geq 4, \quad z \geq \tfrac{1}{4}(x_0 + x_5) + \tfrac{1}{8}\sum_{i=1}^{4} x_i + \tfrac{1}{2}$$

6.3 Mapping to 0–1 Cuts

Assuming domain D_1, the x-cut (39) and z-cut (47) respectively map to 0–1 space as follows:

$$2\sum_{j=1}^{n-1} j(y_{0j} + y_{qj}) + \sum_{i=1}^{q-1}\sum_{j=1}^{n-1} j y_{ij} \geq \tfrac{1}{2}(q+3) \tag{48}$$

$$\sum_{j=0}^{n-1} w_j - 1 \geq \frac{2}{q+3}\sum_{j=1}^{n-1} j(y_{0j} + y_{qj}) + \frac{1}{q+3}\sum_{i=1}^{q-1}\sum_{j=1}^{n-1} j y_{ij} + \tfrac{1}{2} \tag{49}$$

To simplify notation, we suppose in this section that each $S_k = \{k\}$. The arguments are very similar for the more general case. Given this simplification, the 0–1 path model is

$$\begin{array}{lr} \displaystyle\sum_{j=0}^{q} y_{ij} = 1, \;\; i = 0, \ldots, q & (r_i) \\ y_{ij} + y_{i+1,j} \leq w_j, \;\; i = 0, \ldots, q-1, \; j = 0, \ldots, q & (s_{ij}) \\ y_{ij}, w_j \in \{0, 1\} \;\; \text{all } i, j & \end{array} \tag{50}$$

We first observe as follows that the x-cut (48) is redundant. If the w_js are treated as constants equal to 1, (50) describes the feasible set in y_{ij}-space. We show in the following lemma that the equations and inequalities in (50), so modified, describe the convex hull of the feasible set.

Lemma 9 *The coefficient matrix of (50) is totally unimodular when all w_js are set to 1.*

Proof Let A be the coefficient matrix, omitting the rows that correspond to the constraints $y_{ij} \leq 1$. Because the omitted rows have only one non-zero element, it suffices to show that A is totally unimodular. We refer to a row of A corresponding to a constraint (r_i) in (50) as row r_i, and similarly for a row corresponding to a constraint (s_{ij}). It suffices [6] to show that for any subset J of the rows of A, there is a partition J_0, J_1 of J for which every column contains at most one 1 in a row in J_0 and at most one 1 in a row in J_1. This is the case if we define J_0, J_1 inductively as follows. If row $r_1 \in J$, let $r_1 \in J_0$ and $s_{1j} \in J_1$ for all j, and otherwise let $s_{1j} \in J_0$ for all j. Now for any $i > 1$, suppose $s_{i-1,j} \in J_t$ (for all j). If $r_i \in J$, let $r_i \in J_{1-t}$ and $s_{ij} \in J_t$ for all j, and otherwise let $s_{ij} \in J_{1-t}$ for all j. □

Corollary 6 *The x-cut (48) is redundant of the equations and inequalities in (50).*

Proof Because the coefficient matrix of (50) is totally unimodular when w_js are set to 1, the equations and inequalities in (50) describe the convex hull of the feasible set in y_{ij}-space. Thus any valid inequality in variables y_{ij} is redundant of (48) when the w_js are fixed to 1, and therefore redundant for any w_j (since $0 \le w_j \le 1$). In particular, (48) is redundant. □

We cannot use a similar argument to show that the 0–1 z-cut (49) is redundant, because the full model (50) with w_js is not totally unimodular. In fact, the 0–1 z-cut is not redundant, because it is not implied by (50) augmented with symmetry-breaking constraints $w_i \ge w_{i+1}$. The following (extreme point) solution satisfies (50) with $q = 3$ but violates the cut because it results in a left-hand side of $1\frac{2}{7}$ and a right-hand side of $1\frac{5}{14}$.

$$y = \begin{bmatrix} 0 & \frac{4}{7} & 0 & \frac{3}{7} \\ \frac{2}{7} & 0 & \frac{4}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & 0 & \frac{1}{7} \\ 0 & 0 & \frac{4}{7} & \frac{3}{7} \end{bmatrix}, \qquad w = (\tfrac{4}{7}, \tfrac{4}{7}, \tfrac{4}{7}, \tfrac{4}{7})$$

The 0–1 z-cut has no effect on the bound in a problem consisting entirely of a path system, because the relaxation (50) already implies the optimal bound of 2. The sum of the negated constraints (r_{q-1}) and (r_q) with constraints $(s_{q-1,j})$ for $j = 0, \ldots, q$ yields the bound $\sum_j w_j \ge 2$.

7 Computational Results

7.1 Cycles

Table 1 shows LP bounds obtained from several families of cuts in 0–1 space for odd cycles. Each instance is a graph that consist entirely of one odd cycle, parameterized by s and q. All of the overlap sets S_k have size s, and vertex set $V_k = S_k \cup S_{k+1}$ for $k = 1, \ldots, q-1$ (with $V_q = S_q \cup S_1$). For each instance, we solved the linear programming relaxation that minimizes $\sum_j w_j$ subject to (20) and various classes of cuts. The x-cuts and z-cuts are generated only for the given s, and not for smaller overlaps. Because we used a clique cover consisting of maximal cliques, the 0–1 model contains all maximal clique cuts.

The table reflects the fact (Corollary 2) that the two finite-domain cuts, when used together, yield a tighter bound than all odd-hole cuts. In fact, the z-cut alone yields a better bound when $s > 1$. The table also reflects the fact (Lemma 5) that the odd-hole cuts do not improve the bound beyond that obtained from the clique cuts. The finite-domain cuts substantially reduce the integrality gap, sometimes to

Table 1 Lower bounds on the chromatic number in a 0–1 clique formulation of problem instances consisting of one q-cycle with overlap of s.

q	s	Without cuts	Odd-hole cuts only	Lifted cuts	x-cut only	z-cut only	x-cut & z-cut	Optimal value	No. of odd hole cuts
5	1	2.00	2.50	2.50	2.00	2.30	2.60	3	5
	2	4.00	4.00	5.00	4.00	4.50	5.00	5	320
	3	6.00	6.00	7.50	6.00	6.77	7.53	8	3645
	4	8.00	8.00	10.00	8.00	9.00	10.00	10	20,480
	5	10.00	10.00	12.50	10.00	11.26	12.52	13	78,125
7	1	2.00	2.33	2.33	2.00	2.21	2.43	3	7
	2	4.00	4.00	4.67	4.00	4.36	4.71	5	1792
	3	6.00	6.00	7.00	6.00	6.50	7.00	7	45,927
	4	8.00	8.00	9.33	8.00	8.68	9.36	10	458,752
9	1	2.00	2.25	2.25	2.00	2.17	2.33	3	9
	2	4.00	4.00	4.50	4.00	4.28	4.56	5	9216
	3	6.00	6.00	6.75	6.00	6.39	6.78	7	531,441

zero. Thus two finite-domain cuts yield a significantly tighter bound than a large number (ns^q) of odd-hole cuts.

The table also reflects the fact, derived in Section 4, that the finite-domain cuts provide a tighter bound than q lifted odd-hole cuts, except when s is an integer multiple of $(q-1)/2$. The difference in bound is small, but we will see in Section 7 that the finite-domain bound is obtained in much less time in benchmark instances.

Table 2 displays the bounds for the same instances in finite-domain space. The left part of the table shows the bounds $\beta(q,s)/qs+1$ (for an x-cut or z-cut individually) and $2\beta(q,s)/qs+1$ (for both cuts together) derived in Section 4. One might obtain a fairer comparison if clique inequalities are added to the finite-domain model, because they appear in the 0–1 model. In the finite-domain model, clique inequalities correspond to the individual alldiff constraints. We know from [7,17] that for domain D_1, the following is facet-defining for alldiff(X_k):

$$\sum_{i\in V_k} x_i \geq \tfrac{1}{2}|V_k|(|V_k|-1)$$

In the test instances, $|V_k| = 2s$. We therefore added the following cuts:

$$\sum_{i\in V_k} x_i \geq s(2s-1), \quad k = 1,\ldots,q$$

Using Theorem 2, we also added the cuts:

$$z \geq \frac{1}{2s}\sum_{i\in V_k} x_i + s - \tfrac{1}{2}, \quad k = 1,\ldots,q$$

The results appear in the right half of Table 2. The x-cut performs as before, but now the z-cut provides the same bound as in the 0–1 model. When combined, the x-cut and y-cut again deliver the same bound as in the 0–1 model.

Table 2 Lower bounds on the chromatic number in the finite-domain model of problem instances consisting of one q-cycle with overlap of s and color set $\{0, 1, \ldots, n-1\}$.

q	s	No cuts	x-cut only	z-cut only	x-cut & z-cut	Clique cuts	Plus x-cut	Plus z-cut	Plus x- & z-cut
5	1	1.00	1.80	1.80	2.60	1.50	1.80	2.30	2.60
	2	1.00	3.00	3.00	5.00	2.50	3.00	4.50	5.00
	3	1.00	4.27	4.27	7.53	3.50	4.27	6.77	7.53
	4	1.00	5.50	5.50	10.00	4.50	5.50	9.00	10.00
	5	1.00	6.76	6.76	12.52	5.50	6.76	11.26	12.52
7	1	1.00	1.71	1.71	2.43	1.50	1.71	2.21	2.43
	2	1.00	2.86	2.86	4.71	2.50	2.86	4.36	4.71
	3	1.00	4.00	4.00	7.00	3.50	4.00	6.50	7.00
	4	1.00	5.18	5.18	9.36	4.50	5.18	8.68	9.36
9	1	1.00	1.67	1.67	2.33	1.50	1.67	2.17	2.33
	2	1.00	2.78	2.78	4.56	2.50	2.78	4.28	4.56
	3	1.00	3.89	3.89	6.78	3.50	3.89	6.39	6.78

Thus two odd-cycle cuts yield the same bound in the very small finite-domain relaxation (even without clique inequalities) as in the much larger 0–1 relaxation. The finite-domain relaxation contains n variables x_i and $n+2$ constraints, while the 0–1 relaxation contains n^2+n variables y_{ij}, w_j and n^2+n+2 constraints (dropping odd-hole cuts). It may therefore be advantageous to obtain bounds from a finite-domain model rather than a 0–1 model.

7.2 Webs

Table 3 compares bounds obtained for webs $W(q,r)$ when q and $d = q \bmod r$ are coprime. Each instance is a graph that consists entirely of the web $W(q,r)$. We omitted clique cuts (when they exist) because they have no effect on the bound. The table reflects the fact (Corollary 4) that the x-cut and z-cut, when used together, yield a tighter bound than the conventional 0–1 cuts discussed above. The improvement is modest, but only two finite-domain cuts are required, as opposed to n conventional cuts.

7.3 Benchmark Instances

We tested the strength of odd-cycle cuts on benchmark instances of the vertex coloring problem taken from the DIMACS library. Table 4 displays the odd-cycle bounds computed for instances with fewer than 100 variables. Larger instances almost always resulted in an out-of-memory error when the odd-hole cuts were added.

We searched a given graph $G = (V, E)$ for cycles with $s = 1, 2, 3$ using the following greedy algorithm. Let the *co-neighborhood* of a set K of vertices be the intersection of the neighborhoods of the individual vertices in K. For each $s \in \{1, 2, 3\}$ we proceed as follows. We first select the clique $\bar{S}_1$ of size s with the

Table 3 Lower bounds on the chromatic number in a 0–1 formulation of problem instances consisting of a web $W(q, r)$. The bound given by the finite-domain formulation is the same as shown below when both cuts are used.

q	r	Without cuts	0–1 cuts only	x-cut only	z-cut only	x-cut & z-cut	Optimal value	No. of 0–1 cuts
5	2	2	2.50	2.00	2.30	2.60	3	5
7	2	2	3.50	2.29	2.79	3.57	4	7
	3	2	2.33	2.00	2.21	2.43	3	7
9	2	2	4.50	2.78	3.28	4.56	5	9
	4	2	2.25	2.00	2.17	2.33	3	9
10	3	2	3.33	2.20	2.70	3.40	4	10
11	2	2	5.50	3.27	3.77	5.55	6	11
	3	2	3.67	2.36	2.86	3.73	4	11
	4	2	2.75	2.00	2.41	2.82	3	11
	5	2	2.20	2.00	2.14	2.27	3	11

largest co-neighborhood (breaking ties arbitrarily). We then progressively build a path $\bar{S}_1, \bar{S}_2, \ldots$ by adding cliques $\bar{S}_\ell$. For each ℓ, if ℓ is odd, we examine cliques of size s that have vertices in $V \setminus (\bar{S}_1 \cup \cdots \cup \bar{S}_{\ell-1})$ and that continue the path, and select from these a clique $\bar{S}_\ell$ with the largest co-neighborhood. A clique K continues the path if all pairs $(i, j) \in \bar{S}_{\ell-1} \times K$ are edges in E. If ℓ is even, we check, for each clique K that continues the path, whether it allows completion of the cycle; that is, whether each pair $(i, j) \in \bar{S}_1 \times K$ is an edge in E. If so, we generate the cycle. (The vertices in $\bar{S}_1 \cup \cdots \cup \bar{S}_\ell$ may induce edges that are not in a cycle, but the odd-cycle cuts are still valid.) We then let $\bar{S}_\ell$ be a clique that continues the path and has the largest co-neighborhood. The process terminates when no clique continues the path.

Table 4 compares the bounds obtained from the 0–1 model after adding odd-hole cuts for the cycles found with the bounds obtained after adding 0–1 x-cuts and z-cuts for these same cycles. The 0–1 model in these tests is an edge model without clique inequalities, which permits a meaningful comparison of odd-hole and odd-cycle cuts. As noted earlier, odd-hole cuts have little effect in the presence of clique inequalities. Thus each set V_k in the 0–1 model (2) consists of the endpoints of an edge. All odd-hole and odd-cycle cuts are added for each cycle discovered, as opposed to adding only separating cuts. Thus allows us to measure the strength of the cuts without the complicating effect of a separation algorithm.

The results in Table 4 depend on the problem structure, but the finite-domain odd-cycle cuts obtain tighter bounds in most instances, in some cases substantially tighter. As one might expect, the advantage is greater when there are cycles with $s > 1$. The time required to solve the LP relaxation is also consistently less for the finite-domain cuts (because there are only two of them per cycle), in some cases dramatically less. Additionally, the time required to find the cycles is quite small.

Table 4 also indicates the bounds obtained from lifted odd-hole inequalities (27). The bounds are equal to or somewhat weaker than those obtained from odd-cycle cuts, but the time to solve the LP tends to be much longer. The reason for this difference is presumably the number of inequalities, since for each finite-domain cut there are n lifted odd-hole inequalities.

Table 5 shows results of similar tests using a clique-cover 0–1 model rather than an edge model. Here, each set V_k in the 0–1 model (2) consists of the vertices in a clique. Because finding maximal cliques is NP-hard, we used the following heuristic to generate a clique cover C. Starting with $C = \emptyset$, let clique S consist of a single vertex v with the highest positive degree in G. Add to S the vertex with highest degree in $G \setminus S$ that is adjacent to all vertices in S, and repeat until no more additions are possible. At this point, add S to C, remove from G all the edges of the clique induced by S, update the vertex degrees, and repeat the overall procedure until G has no more edges.

The clique-cover model yields bounds that are sometimes weaker than those obtained from the edge model with odd-hole cuts, but often much tighter. Adding odd-cycle cuts to the clique-cover model nonetheless improves the bound in 8 of the 23 instances. Interestingly, the presence of odd-cycle cuts almost always reduces the LP solution time, sometimes substantially. This suggests that it is almost always advantageous to use finite-domain cuts even in a clique-cover model, because they may improve the bound and are very likely to reduce computation time.

8 Conclusion

We explored the idea of obtaining valid inequalities from a finite-domain formulation of a problem that is normally given a 0–1 formulation. We showed that in the case of graph coloring, this approach yields valid inequalities that provide tighter bounds on the chromatic number than known 0–1 cuts for the problem. In particular, we identified facet-defining and other valid inequalities for odd cycles and webs that, when mapped into a 0–1 model, yield a tighter bound than a much larger collection of conventional odd-hole and web cuts.

In addition, finite-domain cuts provide the same tight bound in a relaxation of the finite-domain model as in a relaxation of the 0–1 model. If other families of finite-domain cuts follow this pattern, there could be advantage in obtaining bounds from a finite-domain relaxation that is much smaller than the 0–1 model. Given that some benchmark instances result in 0–1 models that are too large even to load into a linear solver [14], this could provide a viable alternative for solving large graph coloring and related problems.

The alternate polyhedral perspective afforded by the finite-domain formulation therefore seems beneficial, at least in the case of graph coloring. The next step is to seek additional families of finite-domain cuts for graph coloring, perhaps corresponding to combs, anti-webs, and more general structures. The general strategy of obtaining valid inequalities and tight bounds from finite-domain formulations can be investigated for other problem classes.

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Table 4 Lower bounds on the chromatic number in a 0–1 edge formulation of benchmark instances with n vertices and m edges, based on odd-hole cuts, lifted odd-hole cuts, and finite-domain odd-cycle cuts. Number of odd-hole cuts generated and number of cycles found (for $s = 1, 2, 3$) are shown, as are the time in seconds required to find the cycles and solve the LP relaxations.

Instance	n	m	Bound			Opt	No. odd holes	No. cycles found			Cycle heur. time	LP Time (sec)		
			Odd hole	Lifted hole	Odd cycle			$s=1$	$s=2$	$s=3$		Odd hole	Lifted hole	Odd cycle
1-Fulllns_3	30	100	2.00	2.00	2.00	3	540	18	0	0	0.00	0.4	1.0	0.1
1-Fulllns_4	93	593	2.00	2.00	2.00	4	5673	61	0	0	0.09	208.0	23.1	0.4
1-Insertions_4	67	232	1.33	1.33	1.43	4	3216	48	0	0	0.03	30.3	24.8	2.0
2-Fulllns_3	52	201	2.00	2.00	2.00	4	936	18	0	0	0.02	0.9	0.6	0.8
2-Insertions_3	37	72	1.25	1.25	1.33	3	296	8	0	0	0.00	2.9	4.0	0.2
3-Fulllns_3	80	346	2.00	2.00	2.00	5	2000	25	0	0	0.06	25.8	6.4	0.2
3-Insertions_3	56	110	1.20	1.20	1.27	4	560	10	0	0	0.02	11.5	24.1	1.6
4-Insertions_3	79	156	1.17	1.17	1.23	3	948	12	0	0	0.05	12.1	91.6	2.6
david	87	406	2.00	8.00	8.00	10	74211	103	48	10	0.27	11.0	18.9	0.2
huck	74	301	2.00	8.00	8.00	10	29822	71	28	4	0.14	7.2	8.0	0.1
jean	80	254	2.00	8.00	8.00	10	19360	26	0	8	0.17	10.2	21.8	2.1
mug88_1	88	146	2.00	2.00	2.00	3	176	2	0	0	0.08	7.8	38.5	1.8
mug88_25	88	146	2.00	2.00	2.00	3	352	4	0	0	0.07	5.3	36.7	2.1
myciel3	11	20	1.50	1.50	1.60	3	44	4	0	0	0.00	0.0	0.0	0.0
myciel4	23	71	1.50	1.50	1.60	4	414	18	0	0	0.00	0.6	0.8	0.0
myciel5	47	236	1.50	1.50	1.60	5	2115	45	0	0	0.01	7.9	3.6	0.1
myciel6	95	755	1.50	1.50	1.60	6	14535	153	0	0	0.09	1754.7	376.1	0.4
queen5_5	25	160	2.00	2.00	2.00	4	1175	47	0	0	0.00	0.4	0.4	0.1
queen6_6	36	290	2.00	5.00	5.00	6	2628	65	1	0	0.01	1.5	1.3	0.1
queen7_7	49	476	2.00	3.67	3.71	6	11417	105	1	0	0.04	10.6	6.4	0.1
queen8_8	64	728	*	3.33	3.38	8	*	133	5	0	0.08	*	68.5	0.3
queen8_12	96	1368	2.00	8.00	8.00	11	104544	229	31	20	0.41	439.6	272.8	0.9
queen9_9	81	1056	2.00	8.00	8.00	9	69660	193	2	1	0.21	212.4	74.2	0.5

*LP solver ran out of memory.

Table 5 Lower bounds on the chromatic number in a 0–1 clique cover formulation of benchmark instances with n vertices and m edges, based on the clique cover model alone and the clique cover model plus finite-domain odd-cycle cuts. The number of cycles found (for $s = 1, 2, 3$) is shown, as are the time in seconds required to find cliques, find cycles, and solve the LP relaxations.

Instance	n	m	Bound Clique model	Bound +Odd cycle	Opt	No. cycles found $s=1$	$s=2$	$s=3$	Heuristic time (sec) clique	cycle	LP Time (sec) Clique model	+Odd cycle
1-Fulllns_3	30	100	2.00	2.00	3	18	0	0	0.00	0.00	0.78	0.3
1-Fulllns_4	93	593	2.00	2.00	4	61	0	0	0.02	0.09	101.7	3.8
1-Insertions_4	67	232	1.00	1.43	4	48	0	0	0.00	0.03	20.3	2.0
2-Fulllns_3	52	201	3.00	3.00	4	18	0	0	0.00	0.02	10.4	2.5
2-Insertions_3	37	72	1.00	1.33	3	8	0	0	0.00	0.00	0.8	0.1
3-Fulllns_3	80	346	4.00	4.00	5	25	0	0	0.01	0.06	1.7	1.3
3-Insertions_3	56	110	1.00	1.27	4	10	0	0	0.00	0.02	7.2	0.6
4-Insertions_3	79	156	1.00	1.23	3	12	0	0	0.00	0.05	5.1	5.7
david	87	406	10.00	10.00	10	103	48	10	0.00	0.27	9.6	7.6
huck	74	301	10.00	10.00	10	71	28	4	0.00	0.14	6.6	2.8
jean	80	254	8.00	8.00	10	26	0	8	0.00	0.17	39.4	12.5
mug88_1	88	146	2.00	2.00	3	2	0	0	0.00	0.08	18.1	14.0
mug88_25	88	146	2.00	2.00	3	4	0	0	0.00	0.07	33.7	6.9
myciel3	11	20	1.00	1.60	3	4	0	0	0.00	0.00	0.0	0.0
myciel4	23	71	1.00	1.60	4	18	0	0	0.00	0.00	0.2	0.0
myciel5	47	236	1.00	1.60	5	45	0	0	0.00	0.01	1.3	0.1
myciel6	95	755	1.00	1.60	6	153	0	0	0.03	0.09	64.0	0.8
queen5_5	25	160	4.00	4.00	4	47	0	0	0.00	0.00	0.9	1.4
queen6_6	36	290	5.00	5.00	6	65	1	0	0.00	0.01	2.3	0.6
queen7_7	49	476	6.00	6.00	6	105	1	0	0.00	0.04	4.9	5.8
queen8_8	64	728	7.00	7.00	8	133	5	0	0.00	0.08	22.4	16.4
queen8_12	96	1368	11.00	11.00	11	229	31	20	0.00	0.41	168.5	275.2
queen9_9	81	1056	8.00	8.00	9	193	2	1	0.01	0.21	68.4	15.3