Thesis

• Logic and optimization have an underlying unity.
  – Ideas from one area can solve problems in the other.
  – Ideas from both can be applied to data analytics.
Research Programs

- **Apply logic-based methods to optimization.**
  - Decision diagrams.
  - Logic-based Benders decomposition

- **Apply optimization methods to logical inference and data analytics.**
  - Logic-based Benders decomposition.
  - Linear, integer, and nonlinear programming.
Decision Diagrams

- Graphical encoding of a boolean function
  - Historically used for circuit design & verification
  - Adapt to optimization and CP

Hadžić and JH (2006, 2007)
Decision Diagrams

– Collaborators…
    – Henrik Andersen
    – David Bergman*
    – André Ciré**
    – Tarik Hadžić
    – Samid Hoda
    – Willem-Jan van Hoeve
    – Barry O’Sullivan
    – Peter Tiedemann

*2014 Doctoral Thesis Award, Association for CP
**2014 Best Student Paper Award, INFORMS Computing Society
Example:
Stable set problem

Find max-weight subset of nonadjacent vertices

Decision Diagrams
Decision Diagrams

Example:
Stable set problem

Find max-weight subset of nonadjacent vertices

Each path corresponds to a feasible solution.
Example:
Stable set problem

Find max-weight subset of nonadjacent vertices

Longest path corresponds to an optimal solution.
Decision Diagrams

- Key idea: Use relaxed decision diagrams
  - Represent a superset of the feasible set
    - … with a limited-width diagram.
  - First used to strengthen propagation in CP solvers.
    - Reduced 1 million node search trees to 1 node.

Andersen, Hadžić, JH, Tiedemann (2007)
Decision Diagrams

• Key idea: Use **relaxed** decision diagrams
  – Represent a superset of the feasible set
    – … with a **limited-width** diagram.
  – First used to strengthen **propagation** in CP solvers.
    – Reduced 1 million node search trees to 1 node.

  Andersen, Hadžić, JH, Tiedemann (2007)

• Shortest (longest) path in the decision diagram provides a **bound** on optimal value.
  – Leads to general-purpose **optimization** method.

  Bergman, Ciré, van Hoeve, JH (2013)
Decision Diagrams

Exact diagram
Width = 3

Relaxed diagram
Width = 2
Exact DD for stable set problem

To build DD, associate \textbf{state} with each node

\begin{itemize}
  \item $x_j = 0$
  \item $x_j = 1$
\end{itemize}
Exact DD for stable set problem

To build DD, associate **state** with each node
Exact DD for stable set problem

To build DD, associate state with each node
Exact DD for stable set problem

To build DD, associate state with each node
Exact DD for stable set problem

Merge nodes that correspond to the same state
Exact DD for stable set problem

Merge nodes that correspond to the same state
Exact DD for stable set problem

To build DD, associate state with each node
Exact DD for stable set problem

To build DD, associate state with each node
Relaxation Bounding

• To obtain a bound on the objective function:
  – Use a relaxed BDD
  – Analogous to LP relaxation in IP
  – This relaxation is discrete.
  – Doesn’t require the linear inequality formulation of IP.
To build **relaxed** BDD, merge some additional nodes as we go along.
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To build **relaxed** BDD, merge some additional nodes as we go along.
Represents 11 solutions, including 9 feasible solutions.
Width = 2

Represents 11 solutions, including 9 feasible solutions

Longest path gives upper bound on optimal value
Decision Diagrams

- Wider diagrams yield tighter bounds
  - But take longer to build.
  - Adjust width dynamically.
Recursive modeling

- Dynamic programming model of problem, using state variables.
  - Using state variables.
- Rule for merging states to create relaxed DD.
  - Analogous to adding valid inequalities in IP.
- How about curse of dimensionality?
  - Solve by branch and bound in the relaxed diagram, rather than state space enumeration.
Decision Diagrams

• A novel branch-and-bound algorithm.
  – Branch on nodes in last exact layer of relaxed decision diagram.
    – …rather than branch on variables.
    – Create a new relaxed DD rooted at each branching node.
    – Prune search tree using bounds from relaxed DD.
Decision Diagrams

• A novel branch-and-bound algorithm.
  – Branch on nodes in last exact layer of relaxed decision diagram.
    – …rather than branch on variables.
    – Create a new relaxed DD rooted at each branching node.
    – Prune search tree using bounds from relaxed DD.
  – Advantage: a manageable number states may be reachable in first few layers.
    – …even if the state space is exponential.
Decision Diagrams

Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

Diagram is exact down to here

Relaxed decision diagram
Decision Diagrams

Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

Branch on nodes in this layer

Relaxed decision diagram
Decision Diagrams

Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

First branch

New relaxed decision diagram
Decision Diagrams

Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)
Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

Decision Diagrams

Third branch

Continue recursively
Decision Diagrams

• Computational results for optimization.
  – Stable set, max cut, max 2-SAT.
    – Superior to commercial MIP solver (CPLEX) on most instances.
    – Even though the problems have natural MIP models.
    – Obtained best known solution on some max cut instances.
  – Slightly slower than MIP on stable set with precomputed clique cover model, but…
Max cut on a graph

Avg. solution time vs graph density

30 vertices
Max 2-SAT

Performance profile

30 variables

Decision Diagrams

Number of instances solved

Computation time (sec)

MDDs

CPLEX
Decision Diagrams

Max 2-SAT

Performance profile

40 variables

Number of instances solved vs. Computation time (sec)

- MDDs
- CPLEX
Decision Diagrams

• Potential to scale up
  – No need to load large inequality model into solver.
  – Parallelizes very effectively
    – Near-linear speedup.
    – Much better than mixed integer programming.
Decision Diagrams

• Next steps
  – Decision diagram technology is ready for large-scale applications from industry
    – …which will also help develop the technology.
Decision Diagrams

• Next steps
  – Decision diagram technology is ready for large-scale applications from industry
    – …which will also help develop the technology.
  – Give problems a dynamic programming model.
    – …natural for many applications.
    – But solve by branch and bound, not DP.
  – Extend to stochastic DP and optimal control.
    – Current research.
Logic-Based Benders

• **Logic-based Benders decomposition** is a generalization of classical Benders decomposition.


• Solves a **projection** problem.
  – ...and therefore a **logical inference** problem.
Logic-Based Benders

– Collaborators…
  – André Ciré
  – Elvin Çoban
  – Aliza Heching
  – Greger Ottosson
  – Erlendur Thorsteinsson*
  – Hong Yan

*2001 Best Paper Award, CP conference
Logic-Based Benders

- Subproblem is an **arbitrary optimization** problem.
  - In classical Benders, subproblem must be linear or nonlinear programming problem.
    - LBBD replaces LP dual with **inference dual**.
  - Solves problems of the form
    $$\min f(x, y)$$
    $$(x, y) \in S$$
    $$x \in D$$
Logic-Based Benders

- Decompose problem into master and subproblem.
  - Subproblem is obtained by fixing $x$ to solution value in master problem.

Master problem

$$\min z$$
$$z \geq g_k(x) \quad \text{(Benders cuts)}$$
$$x \in D$$

Find $x$ that minimizes cost $z$ subject to Benders cuts obtained from solutions in previous iterations $k$.

Subproblem

$$\min f(\bar{x}, y)$$
$$\quad (\bar{x}, y) \in S$$

Obtain proof of optimality \textit{(inference dual)}. Use same proof to deduce cost bounds for other assignments, yielding Benders cut.

Trial value $\bar{x}$ that solves master

Benders cut $z \geq g_k(x)$
Logic-Based Benders

- Iterate until master problem value equals best subproblem value so far.
  - This yields optimal solution.

Master problem

\[
\min z \\
z \geq g_k(x) \quad \text{(Benders cuts)} \\
x \in D
\]

Find \( x \) that minimizes cost \( z \) subject to Benders cuts obtained from solutions in previous iterations \( k \).

Subproblem

\[
\min f(\bar{x}, y) \\
(\bar{x}, y) \in S
\]

Trial value \( \bar{x} \) that solves master

Benders cut

\[
z \geq g_k(x)
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Obtain proof of optimality (inference dual). Use same proof to deduce cost bounds for other assignments, yielding Benders cut.
Logic-Based Benders

• Substantial speedup for many applications.
  – Several orders of magnitude relative to state of the art.
Logic-Based Benders

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• Some applications:
  - Circuit verification
  - Chemical batch processing (BASF, etc.)
  - Steel production scheduling
  - Auto assembly line management (Peugeot-Citroën)
  - Automated guided vehicles in flexible manufacturing
  - Allocation and scheduling of multicore processors (IBM, Toshiba, Sony)
  - Facility location-allocation
  - Stochastic facility location and fleet management
  - Capacity and distance-constrained plant location
Logic-Based Benders

- Some applications…
  - Transportation network design
  - Traffic diversion around blocked routes
  - Worker assignment in a queuing environment
  - Single- and multiple-machine allocation and scheduling
  - Permutation flow shop scheduling with time lags
  - Resource-constrained scheduling
  - Wireless local area network design
  - Service restoration in a network
  - Optimal control of dynamical systems
  - Sports scheduling
Logic-Based Benders

- Application to planning and scheduling.
  - Assign tasks in master, schedule them in subproblem.
  - Combine **mixed integer programming (MIP)** and **CP**

Master problem

Assign tasks to resources to minimize cost.
Solve by **MIP**.

Subproblem

Schedule tasks on each resource, subject to time windows.

**CP** obtains proof of optimality (inference dual).

Use **same proof** to deduce cost for other assignments, yielding **Benders cut**.

\[ z \geq g_k(x) \]
Logic-Based Benders

• **Objective function**
  
  – Cost is based on **task assignment only**.

  \[
  \text{cost} = \sum_{ij} c_{ij} x_{ij}, \quad x_{ij} = 1 \text{ if task } j \text{ assigned to resource } i
  \]

  – So cost appears only in the **master problem**.
  – Scheduling subproblem is a **feasibility problem**.
Logic-Based Benders

- **Objective function**
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    \]
    
    - So cost appears only in the **master problem**.
    - Scheduling subproblem is a **feasibility problem**.

- **Benders cuts**
  - They have the form \( \sum_{j \in J_i} (1 - x_{ij}) \geq 1, \text{ all } i \)
    
    - where \( J_i \) is a set of tasks that create infeasibility when assigned to resource \( i \).
Logic-Based Benders

- Resulting Benders decomposition:

Master problem

\[
\min \ z = \sum_{ij} c_{ij} x_{ij}
\]

Benders cuts

Subproblem

Schedule jobs on each resource.

**CP** may obtain proof of infeasibility on some resources (inference dual).

Use **same proof** to deduce infeasibility for some other assignments, yielding **Benders cut**.

\[
\sum_{j \in J_i} (1 - x_{ij}) \geq 1, \quad \text{for infeasible resources } i
\]
Performance profile

180 instances

Designed to be **hard** for LBBDD

MIP is faster than CP

Ciré, Çoban, JH (2015)
Performance profile

50 instances

More realistic

Ciré, Çoban, JH (2015)
Logic-Based Benders

• **Current research**
  – Apply to **robust** scheduling.
  – Use a **decision diagram** in the master problem.
  – Apply to **logical inference** from large datasets.
Logic-Based Benders

• Robust scheduling
  – …with uncertainty sets.
    – Uncertainty subproblem becomes Benders subproblem.
  – IT service center scheduling.
    – Projects subject to uncertain delays
    – Give customers a reasonable worst-case completion date.

Çoban, Heching, JH (2015)
Logic-Based Benders

- **Decision diagram** in master problem.
  - Add Benders cuts by **modifying** the decision diagram.
    - Requires **separation algorithm** for decision diagrams
  - Home healthcare scheduling.
    - Master problem assigns healthcare aides to patients.
    - Subproblem schedules visits and routes the aides.
  - **Key question:**
    - How much will the decision diagram **grow** as Benders cuts are added?

Ciré, JH (2014)

Heching, JH (2015)
Growth of Separating Diagram for All but 3 Instances
Growth of Separating Diagram for 2 Harder Instances
Logic-Based Benders

• Logical inference.
  – Benders cuts describe the projection of the feasible set onto the master problem.
  – This is the fundamental logical inference problem.
    – Original application was to logic circuit verification.
Logical Inference

• Connection between optimization and logic.
  – Both are projection problems.
  – Projection = extract information involving a subset of variables.
Logical Inference

• Connection between optimization and logic.
  – Both are projection problems.
  – Projection = extract information involving a subset of variables.

• Optimization…
  – Project feasible set onto a single variable representing the objective function.

• Inference…
  – Project knowledge base onto a subset of propositional variables.
Logical Inference

• Example:
  – Knowledge base consists of logical clauses.
  – Derive all implications involving $x_1$, $x_2$.
  – This is a projection problem.

\[
\begin{align*}
X_1 \lor X_2 \\
\neg X_1 \lor X_3 \\
\neg X_1 \lor \neg X_2 \lor \neg X_3 \\
X_1 \lor X_3 \lor X_4 \\
X_2 \lor X_3 \lor \neg X_4 \\
X_1 \lor X_2 \\
\neg X_1 \lor \neg X_2
\end{align*}
\]
Logical Inference

• Example:
  – Knowledge base consists of logical clauses.
  – Derive all implications involving $x_1$, $x_2$.
  – This is a projection problem.

• Solved by logic-based Benders.
  – Clauses in the projection are Benders cuts
    – They are also conflict clauses containing $x_1$, $x_2$ from a SAT algorithm.

\[
\begin{align*}
X_1 \lor X_2 \\
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\neg X_1 \lor \neg X_2 \lor \neg X_3 \\
X_1 \lor X_3 \lor X_4 \\
X_2 \lor X_3 \lor \neg X_4 \\
\end{align*}
\]

Projection
Logical Inference

• A central problem today:
  – Associate each inference with its **probability**, **relevance**, or **confidence**.
    – For example, IBM’s **Watson** (Jeopardy player).
Logical Inference

Watson technology first applied to medicine (WatsonPaths).

Draws inferences from medical literature and clinical guidelines.

About 1 million articles listed per year in PubMed.

Probably 1.5-2 million overall.
Logical Inference

• The problem was first posed for probability...
Logical Inference

• The problem was first posed for **probability**...
  – …by George Boole.
    – He viewed **probability logic** as a key contribution.
Logical Inference

• The problem was first posed for probability...
  – …by George Boole.
    – He viewed probability logic as his most important contribution.
    – Formulated inference problem as linear programming.
    – Can be solved by modern column generation methods.
### Logical Inference

#### Example

<table>
<thead>
<tr>
<th>Clause</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\overline{x}_1 \lor x_2$</td>
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Deduce probability range for $x_3$
Logical Inference

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Deduce probability range for $x_3$

Linear programming model

\[
\begin{bmatrix}
01010101 \\
00001111 \\
11110011 \\
11011101 \\
11111111
\end{bmatrix} \begin{bmatrix}
p_{000} \\
p_{001} \\
p_{010} \\
p_{101} \\
p_{111}
\end{bmatrix} = \begin{bmatrix}
\pi_0 \\
0.9 \\
0.8 \\
0.4 \\
1
\end{bmatrix}
\]

\[p_{000} = \text{probability that } (x_1, x_2, x_3) = (0,0,0)\]
Logical Inference

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$p_{000} = \text{probability that} \ (x_1, x_2, x_3) = (0,0,0)$

Solution: $\pi_0 \in [0.1,0.4]$
Logical Inference

• Next steps
  – Apply **large-scale optimization** methods to logical inference.
    – They are substantially **underutilized** for this purpose.
  – In particular:
    – Use **logic-based Benders** for propositional logic, etc.
    – Use **column generation** for probability logic.
    – Use **linear programming** for belief logics, relevance, and variations of Dempster-Shafer theory.
    – Use **nonlinear programming** for Bayesian networks.
    – Use **decision diagrams** for queries and what-if analysis.

Chandru, JH (1999)