Logic, Optimization and Data Analytics

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Thesis

- **Logic** and **optimization** have an **underlying unity.**
  - Ideas from one area can solve problems in the other.
  - Ideas from both can be applied to **data analytics**.

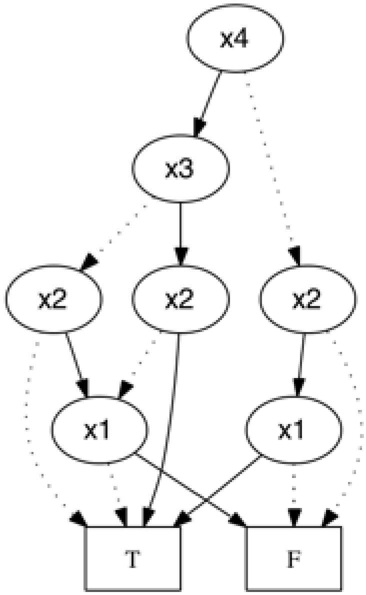
Research Programs

- Apply **logic-based methods** to **optimization**.
  - Decision diagrams.
  - Logic-based Benders decomposition
- Apply **optimization methods** to **logical inference** and **data analytics**.
  - Logic-based Benders decomposition.
  - Linear, integer, and nonlinear programming.

Decision Diagrams

- Graphical encoding of a boolean function
  - Historically used for circuit design & verification
  - Adapt to optimization and CP

Hadžić and JH (2006, 2007)



Decision Diagrams

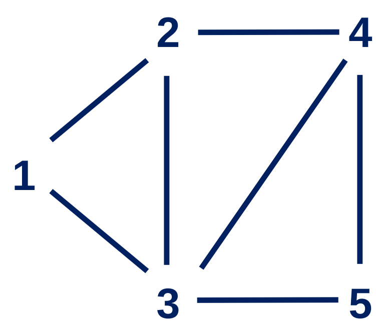
- Collaborators…
  - Henrik Andersen
  - David Bergman*
  - André Ciré**
  - Tarik Hadžić
  - Samid Hoda
  - Willem-Jan van Hoeve
  - Barry O’Sullivan
  - Peter Tiedemann

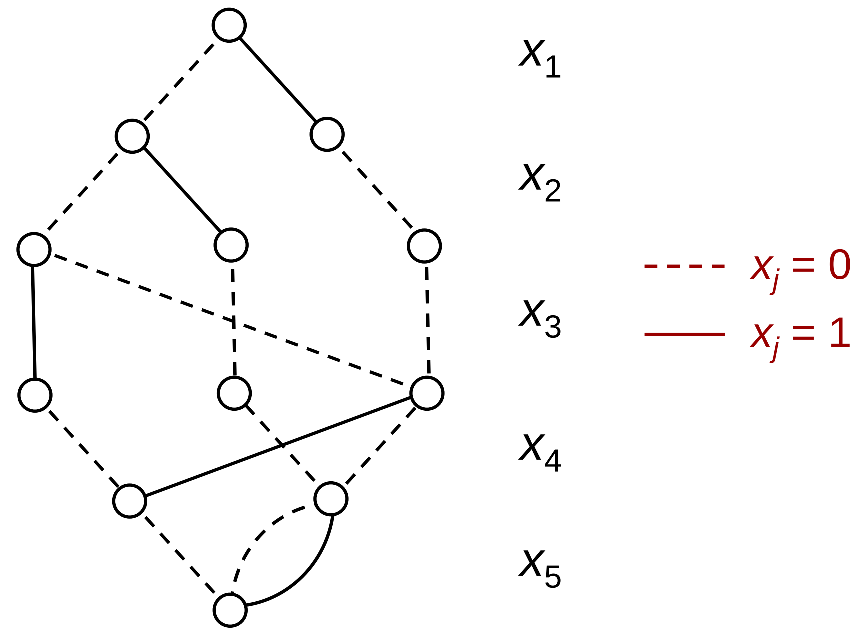
*****2014 *Doctoral Thesis Award*, Association for CP
**2014 *Best Student Paper Award,* INFORMS Computing Society

Decision Diagrams

Example:
Stable set problem

Find max-weight subset
of nonadjacent vertices





Decision Diagrams

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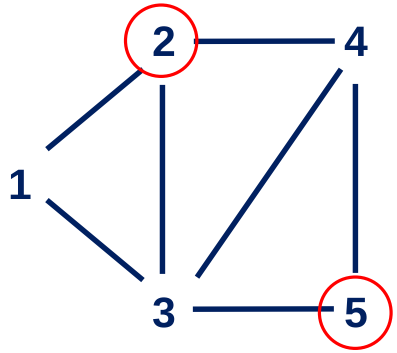

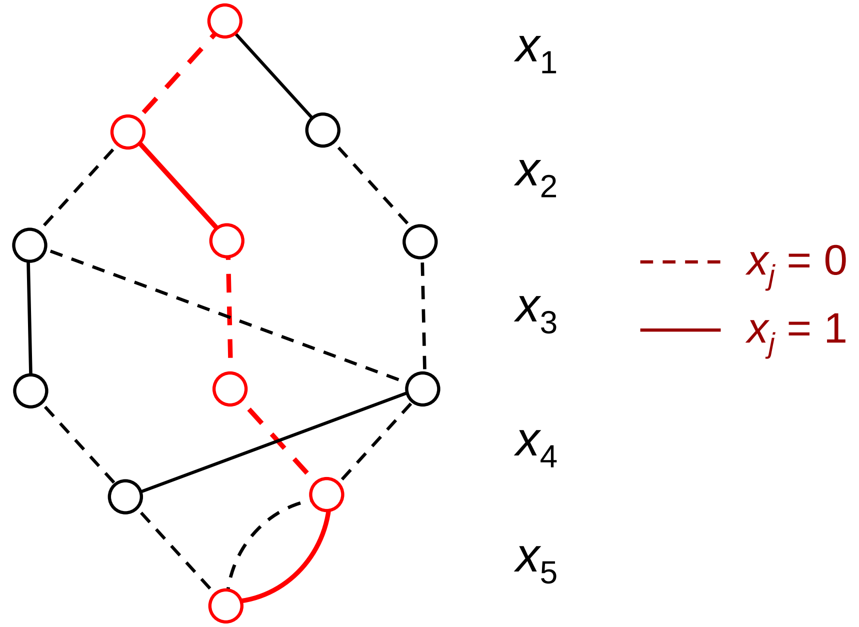

Each path corresponds to a feasible solution.

Decision Diagrams

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Stable set problem

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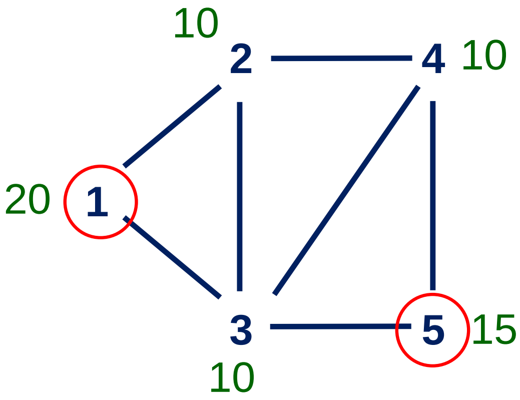

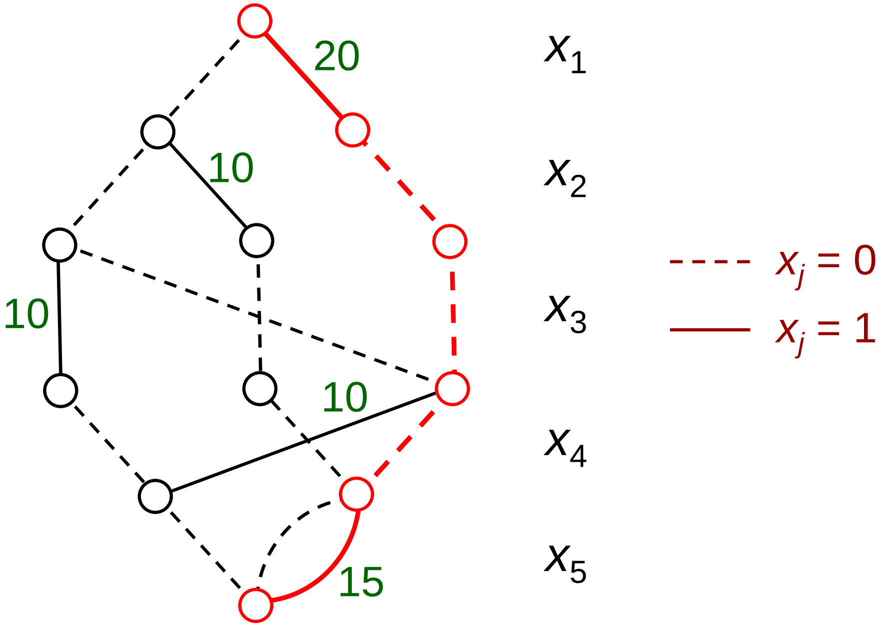

Longest path corresponds to an optimal solution.

Decision Diagrams

- Key idea: Use **relaxed** decision diagrams
  - Represent a superset of the feasible set
    - … with a **limited-width** diagram.
  - First used to strengthen **propagation** in CP solvers.
    - Reduced 1 million node search trees to 1 node.

Andersen, Hadžić, JH, Tiedemann (2007)

Decision Diagrams

- Key idea: Use **relaxed** decision diagrams
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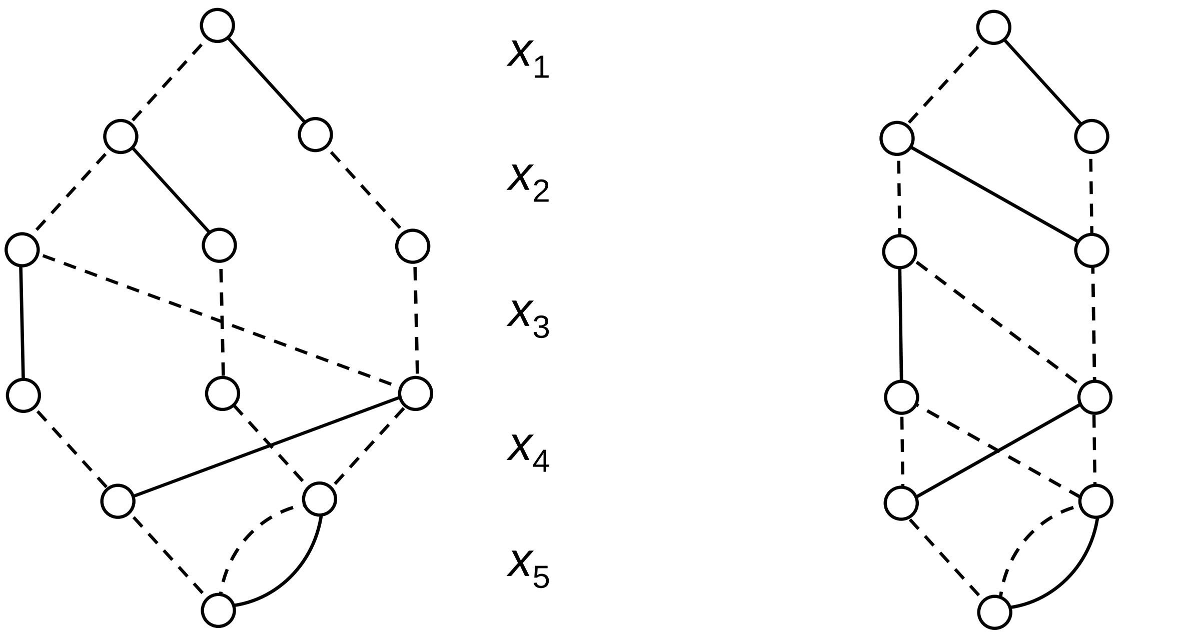
- Shortest (longest) path in the decision diagram provides a **bound** on optimal value.
  - Leads to general-purpose **optimization** method.

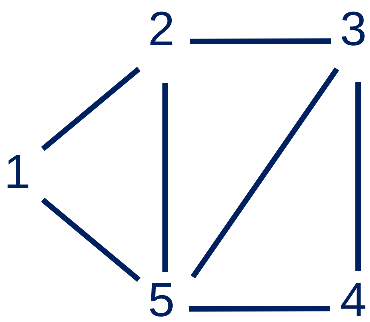
Bergman, Ciré, van Hoeve, JH (2013)

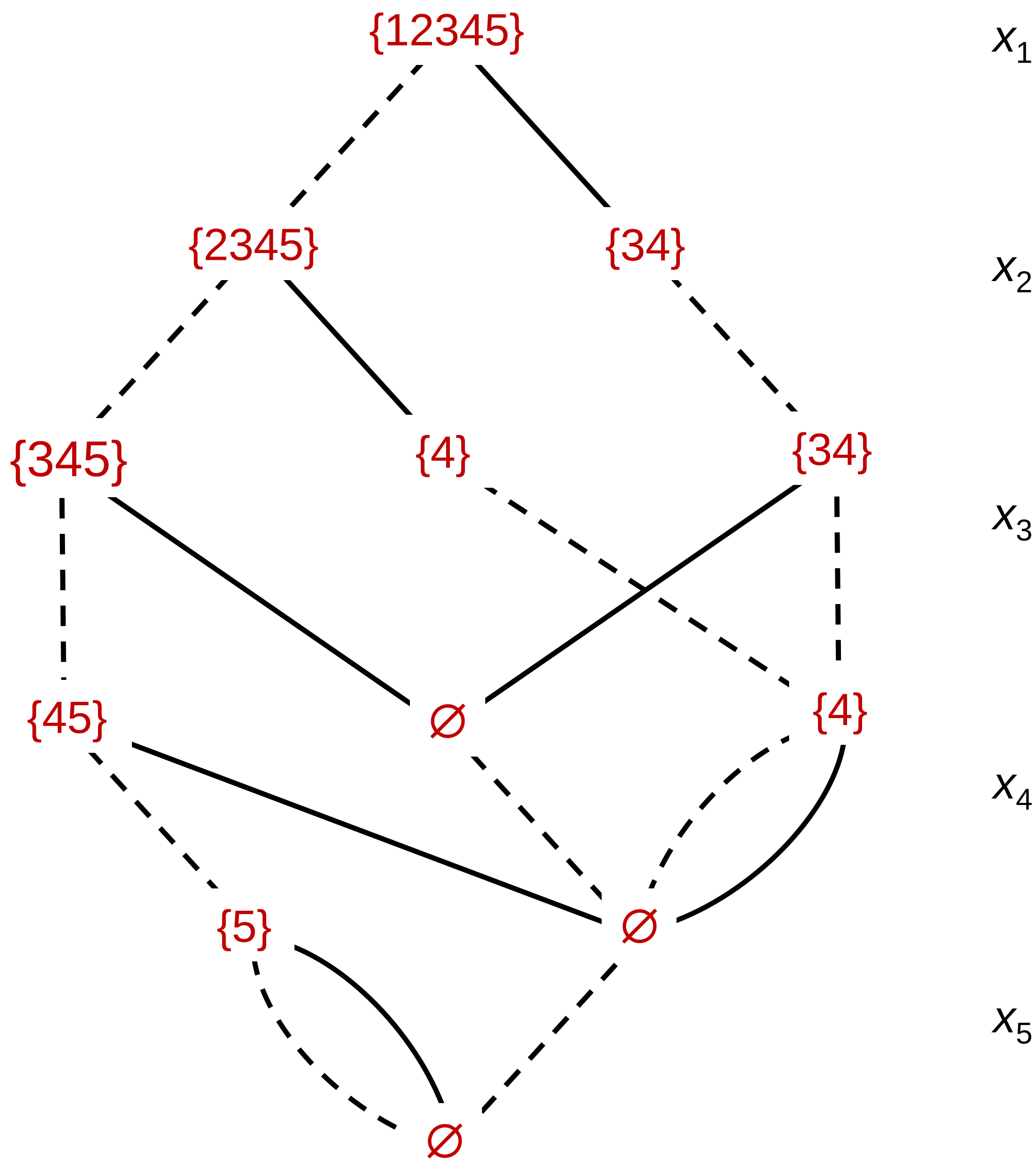
Decision Diagrams



Exact diagram
Width = 3

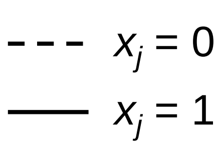
Relaxed diagram
Width = 2

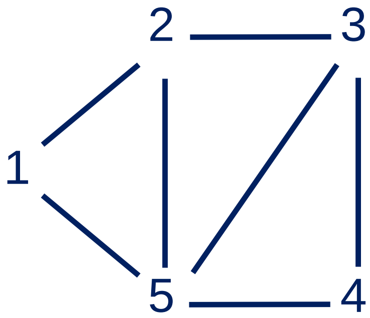




Exact DD for
stable set problem

To build DD,
associate **state**
with each node





{12345}

x_1

x_2

Exact DD for
stable set problem

x_3

x_4

To build DD,
associate **state**
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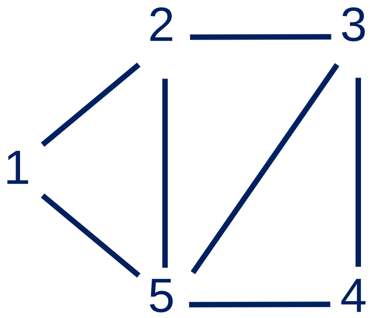
x_5



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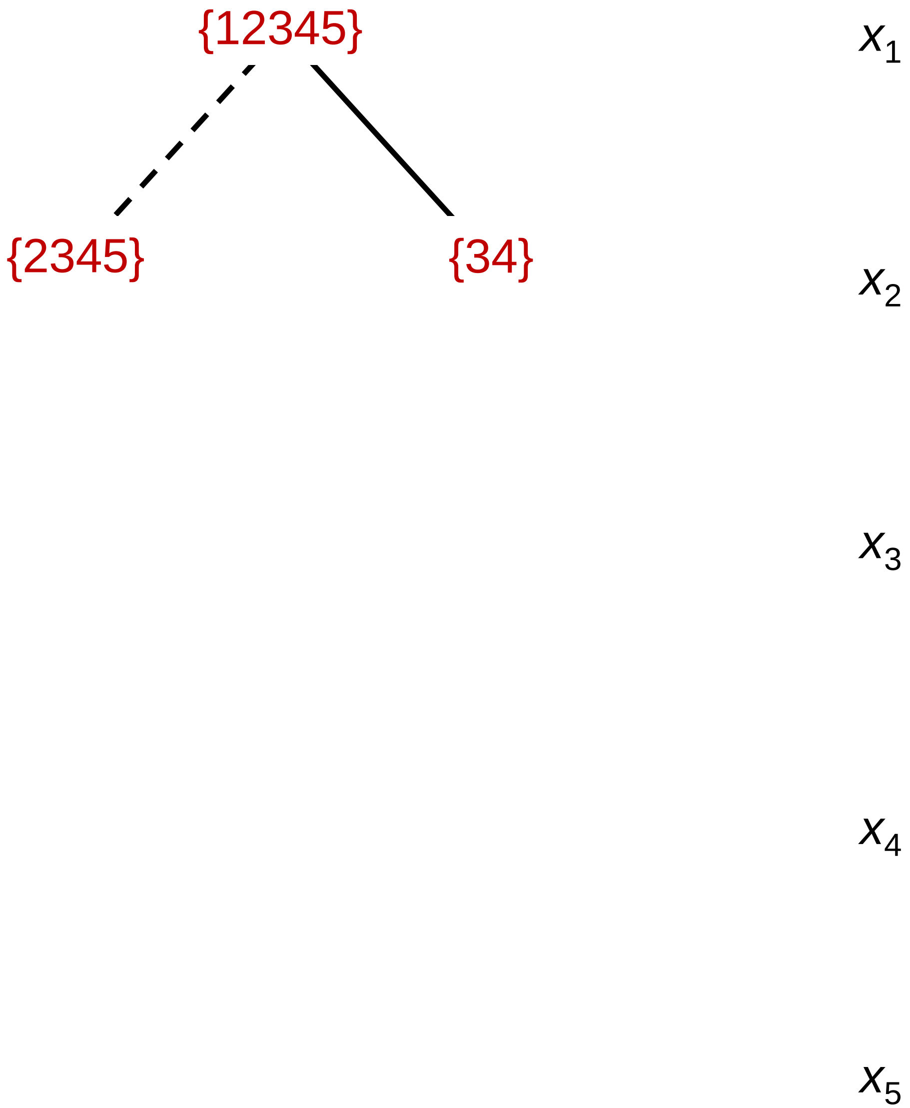

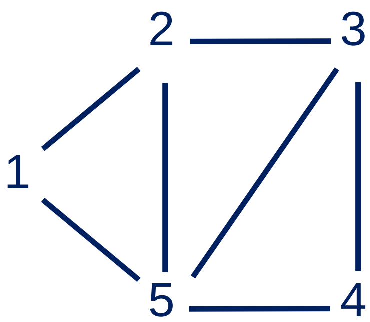

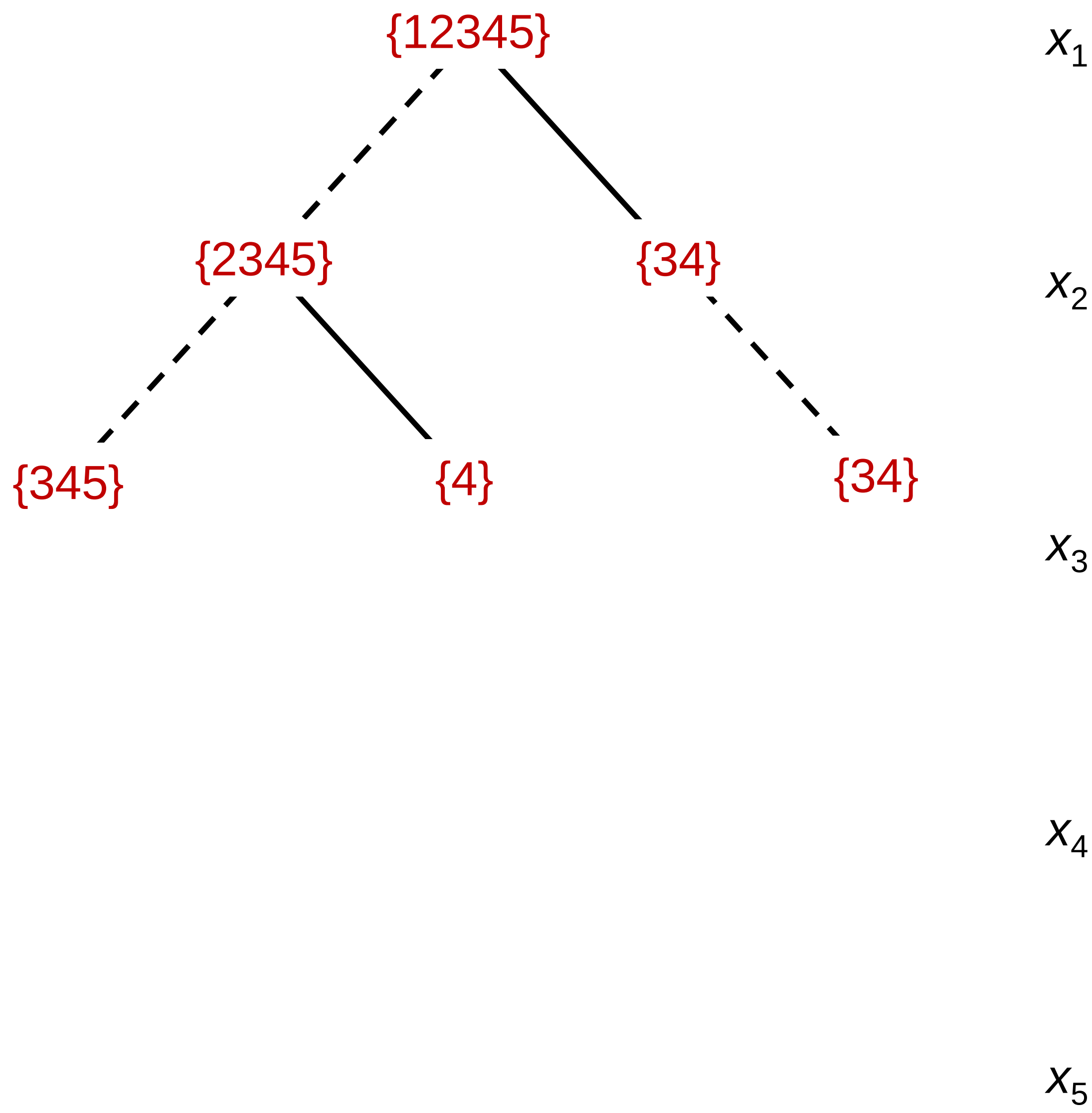

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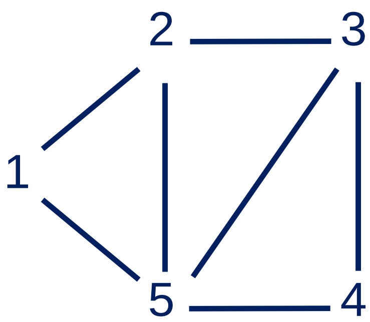
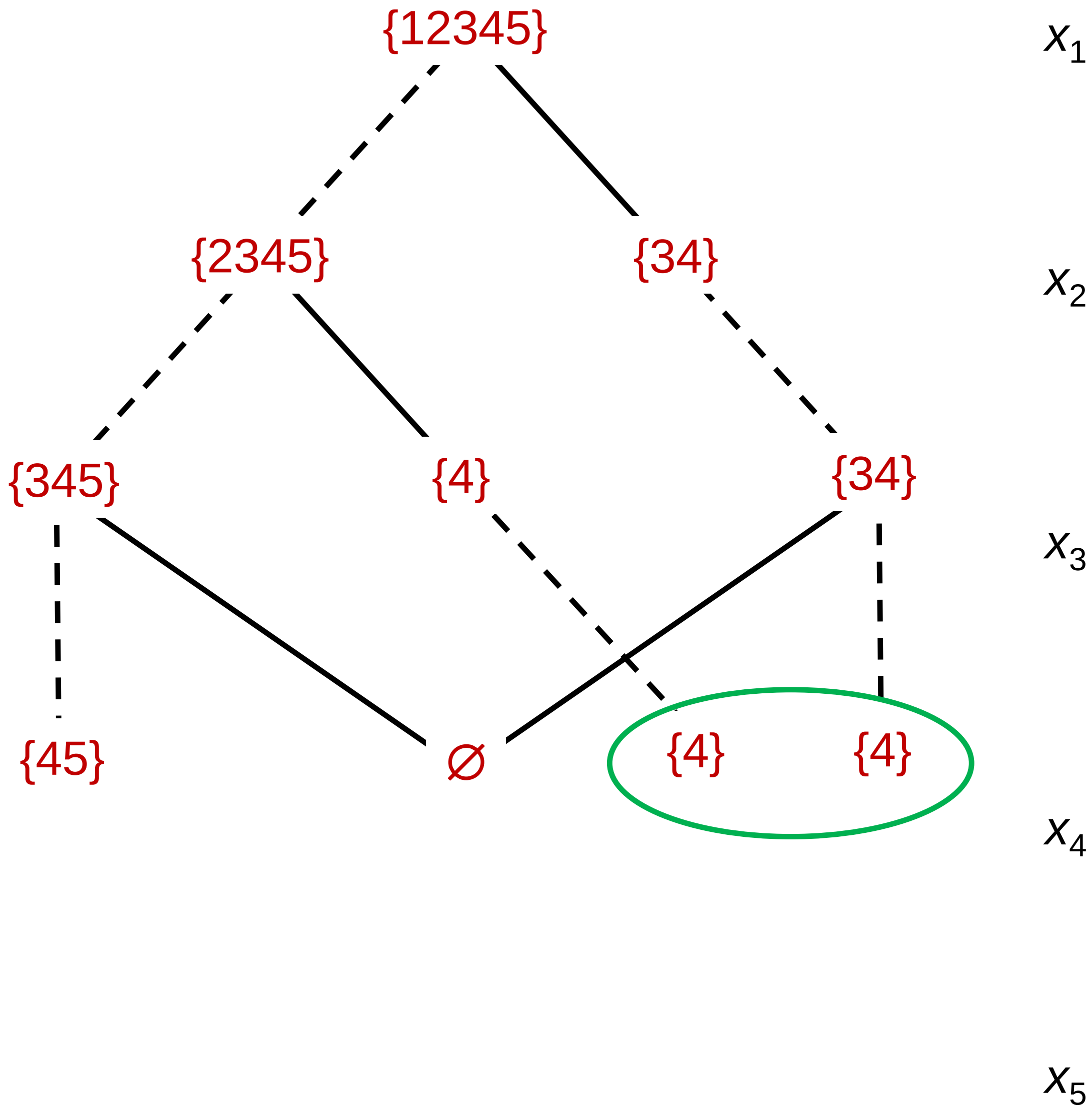
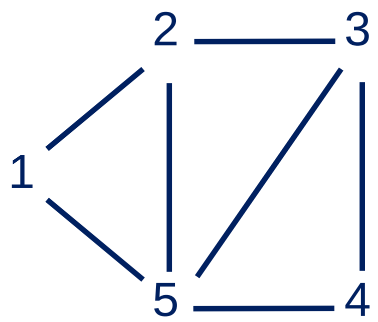
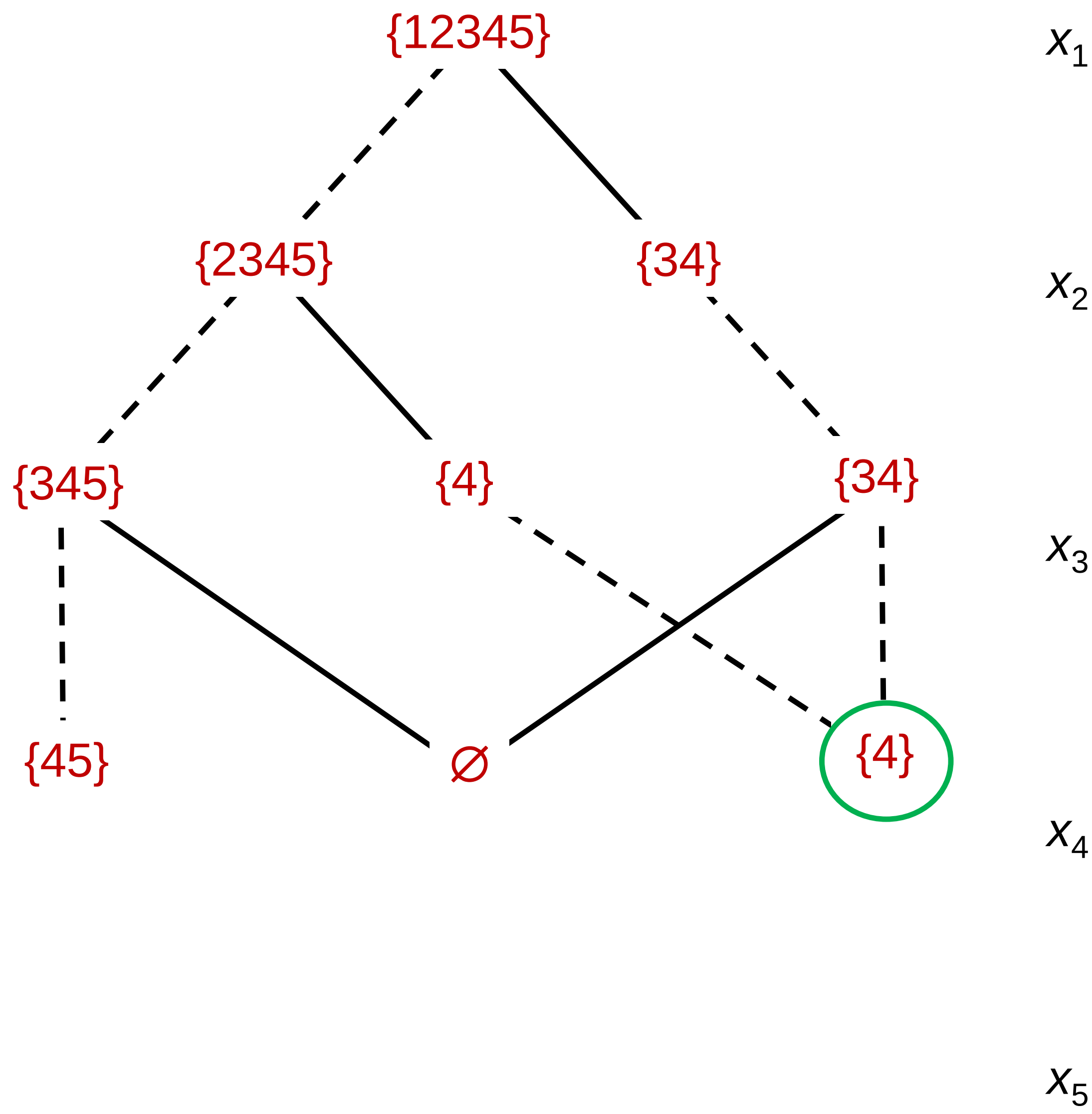




Exact DD for stable set problem

Merge nodes that correspond to the same state





Exact DD for
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Merge nodes that
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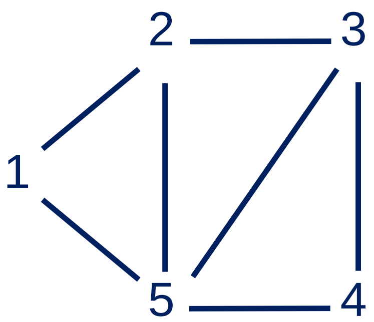

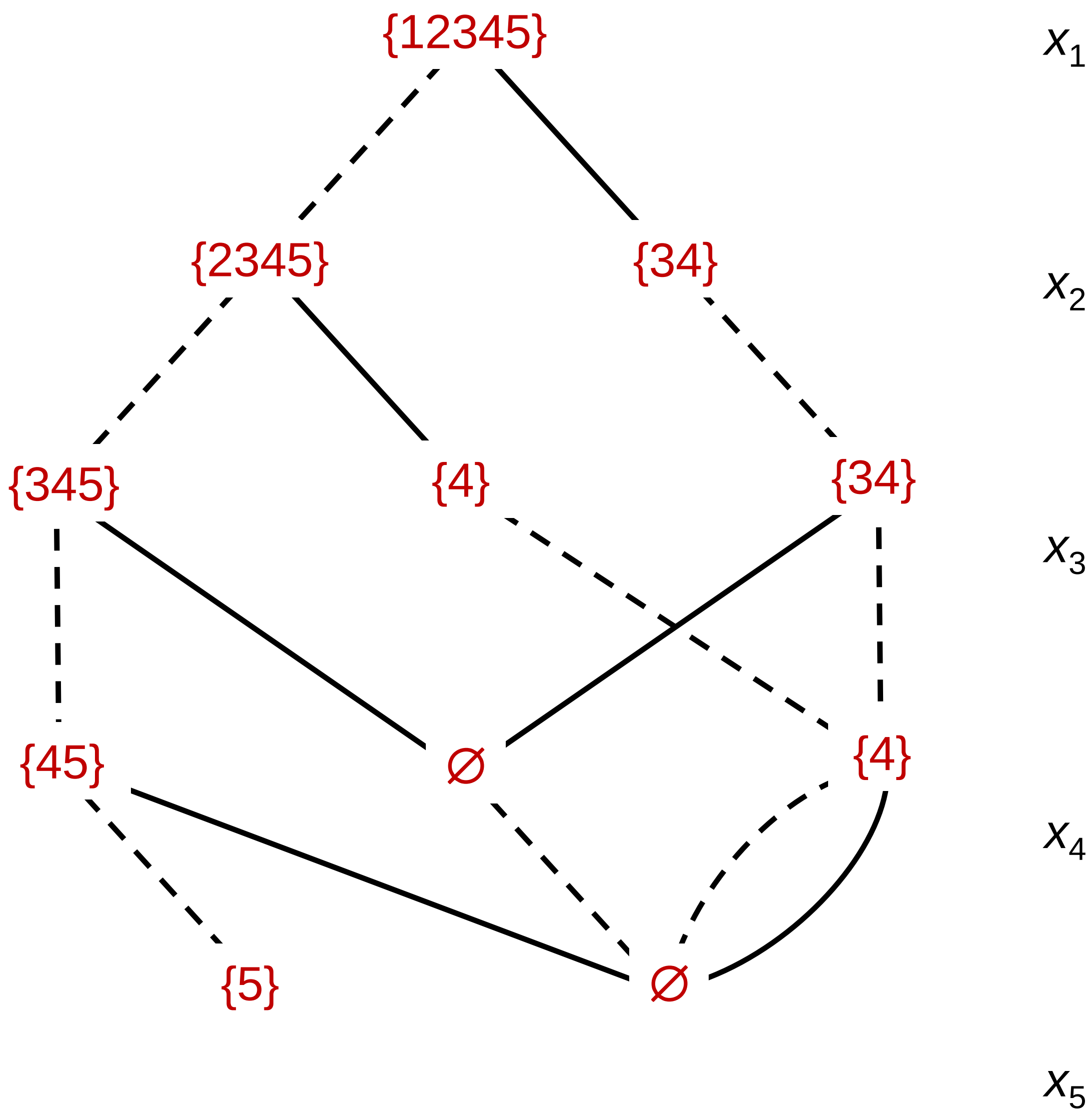

Exact DD for
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To build DD,
associate **state**
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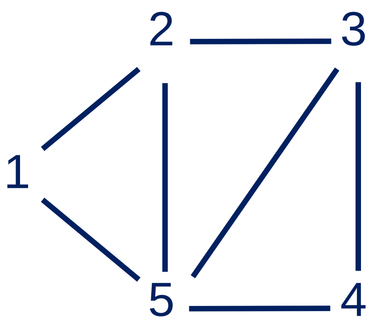

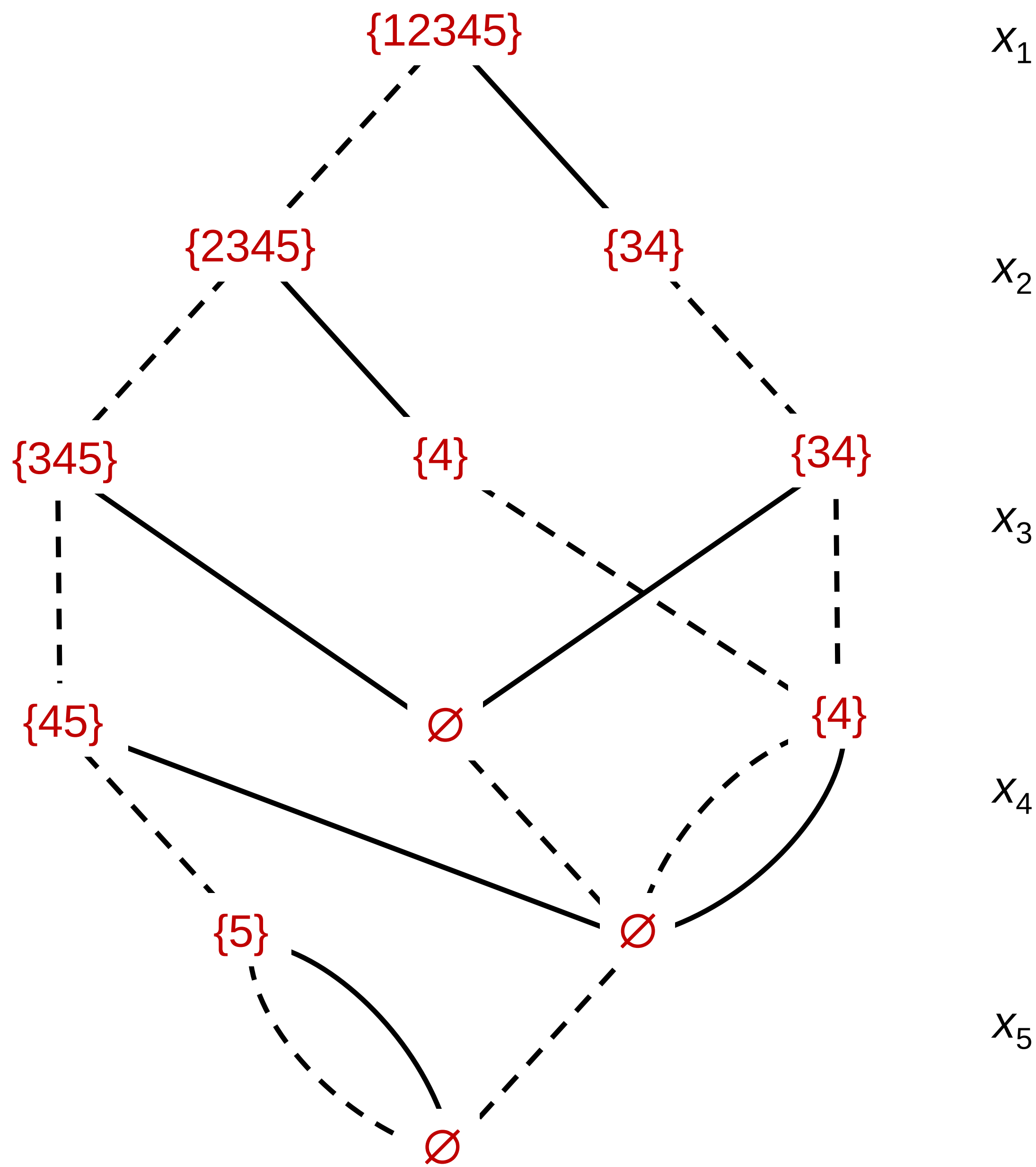

Exact DD for
stable set problem

To build DD,
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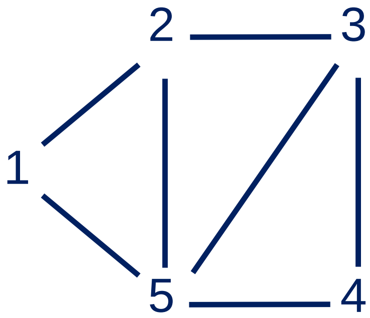
Relaxation Bounding

- To obtain a bound on the objective function:
  - Use a **relaxed** BDD
  - Analogous to LP relaxation in IP
  - This relaxation is **discrete**.
  - Doesn’t require the linear inequality formulation of IP.



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x_1

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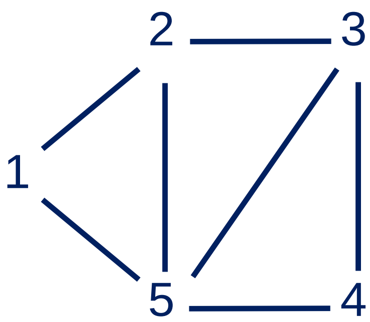
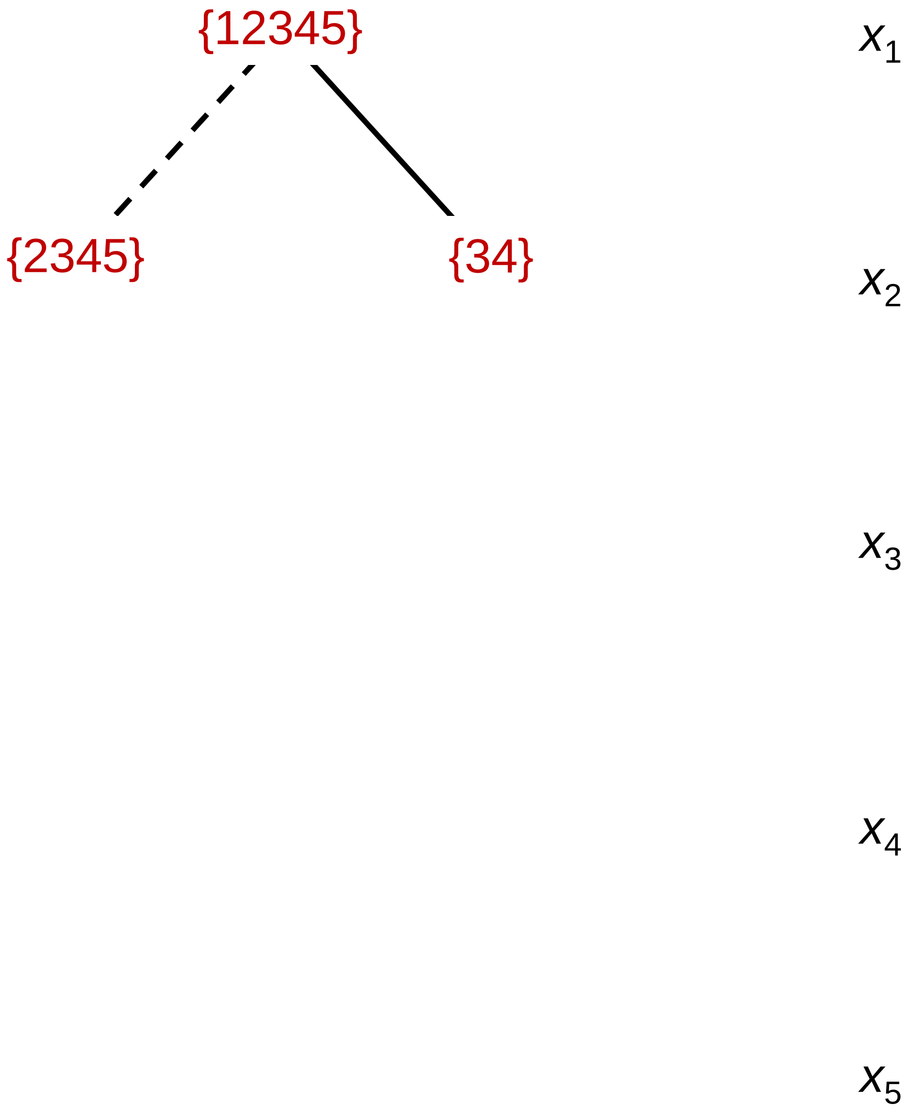
x_3

x_4

x_5

To build **relaxed** BDD, merge some additional nodes as we go along





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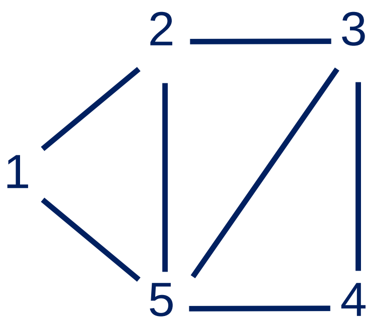

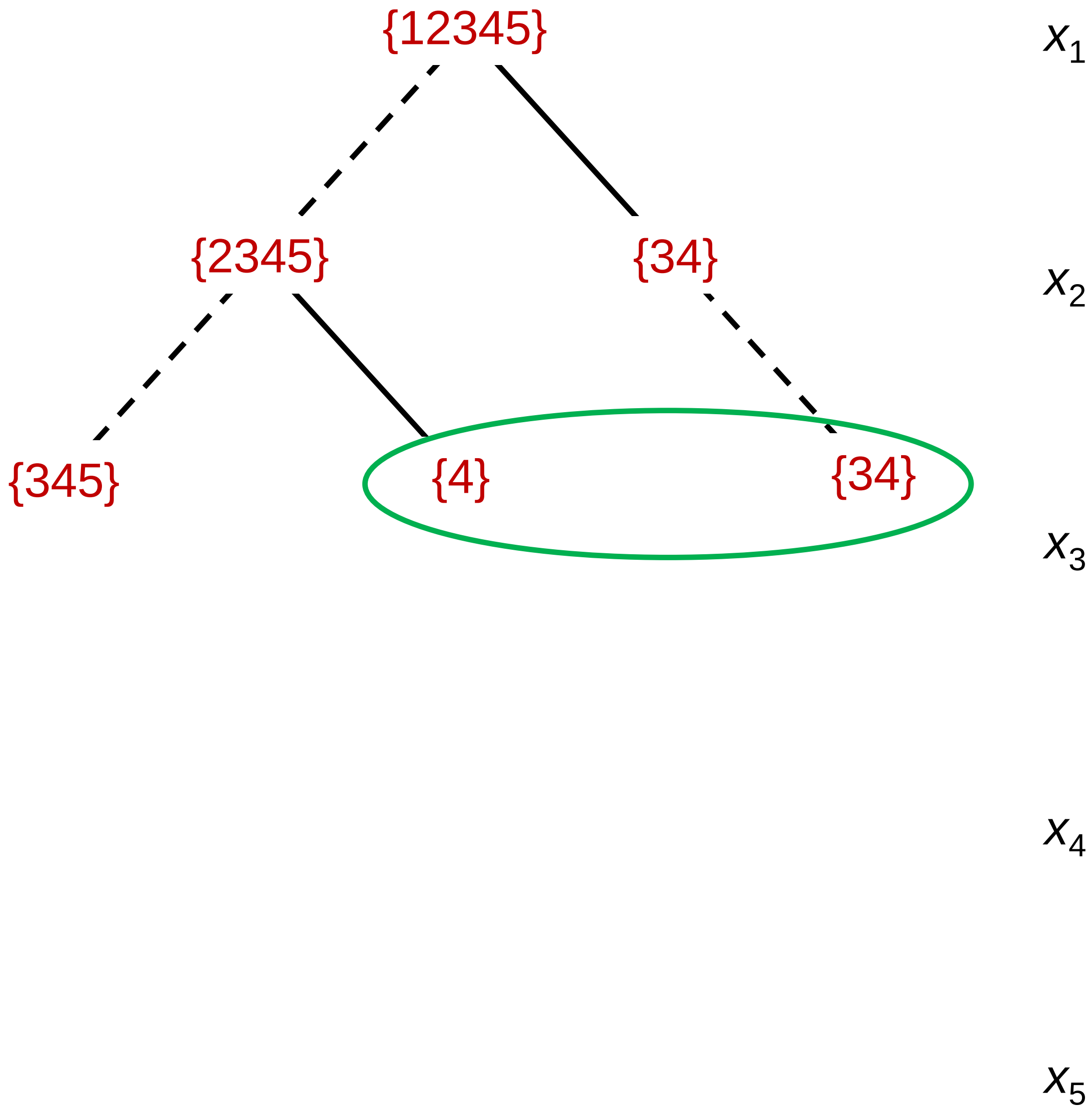

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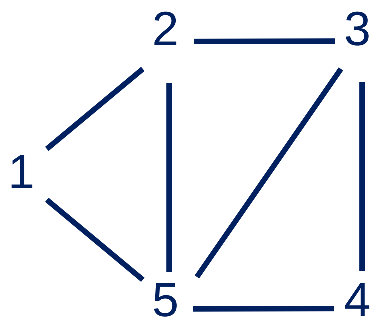

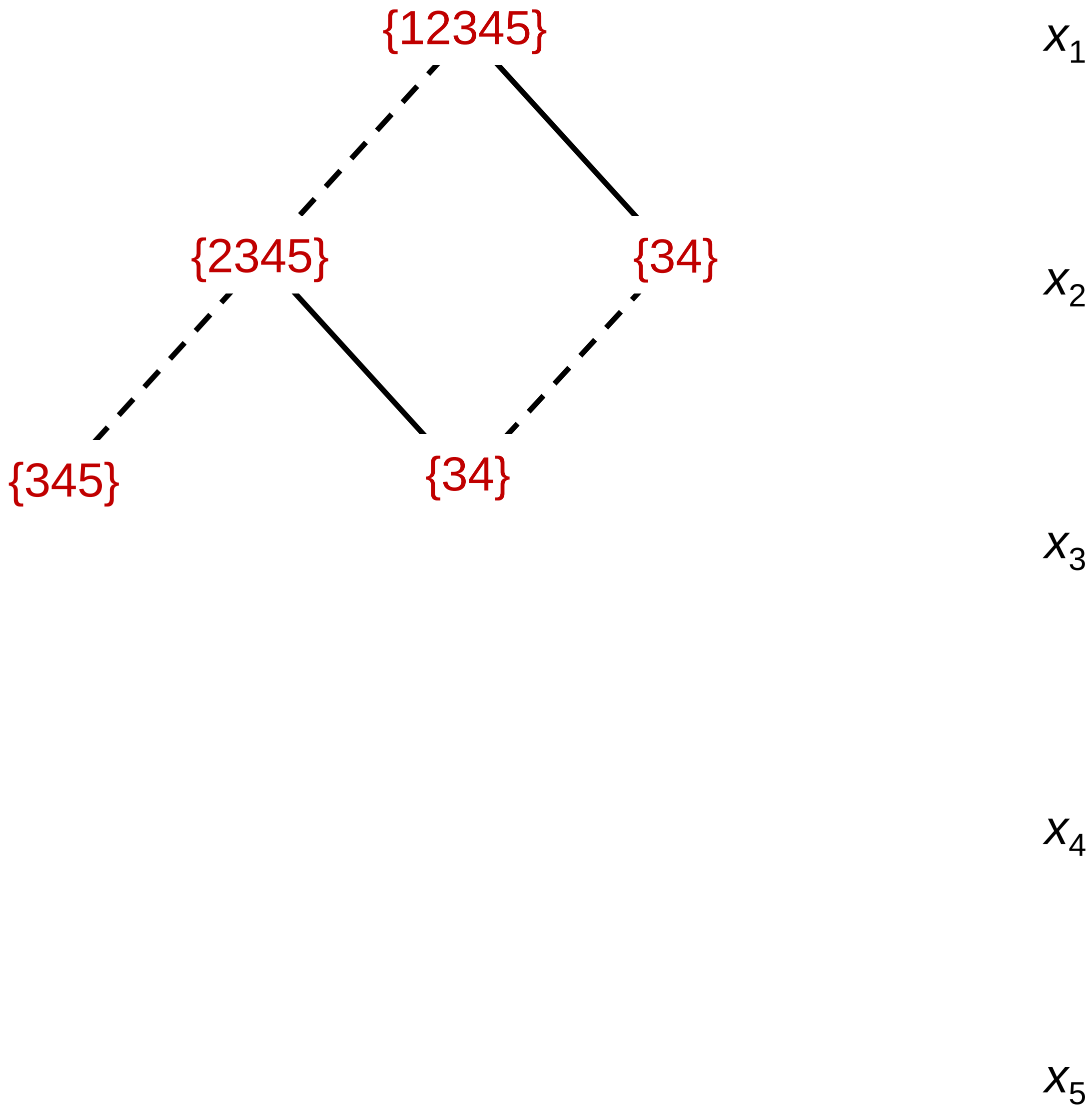

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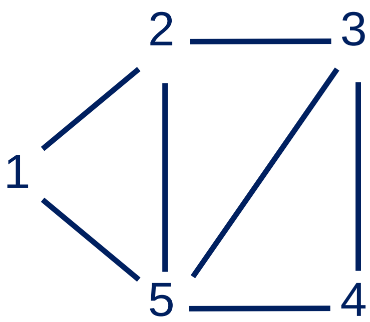

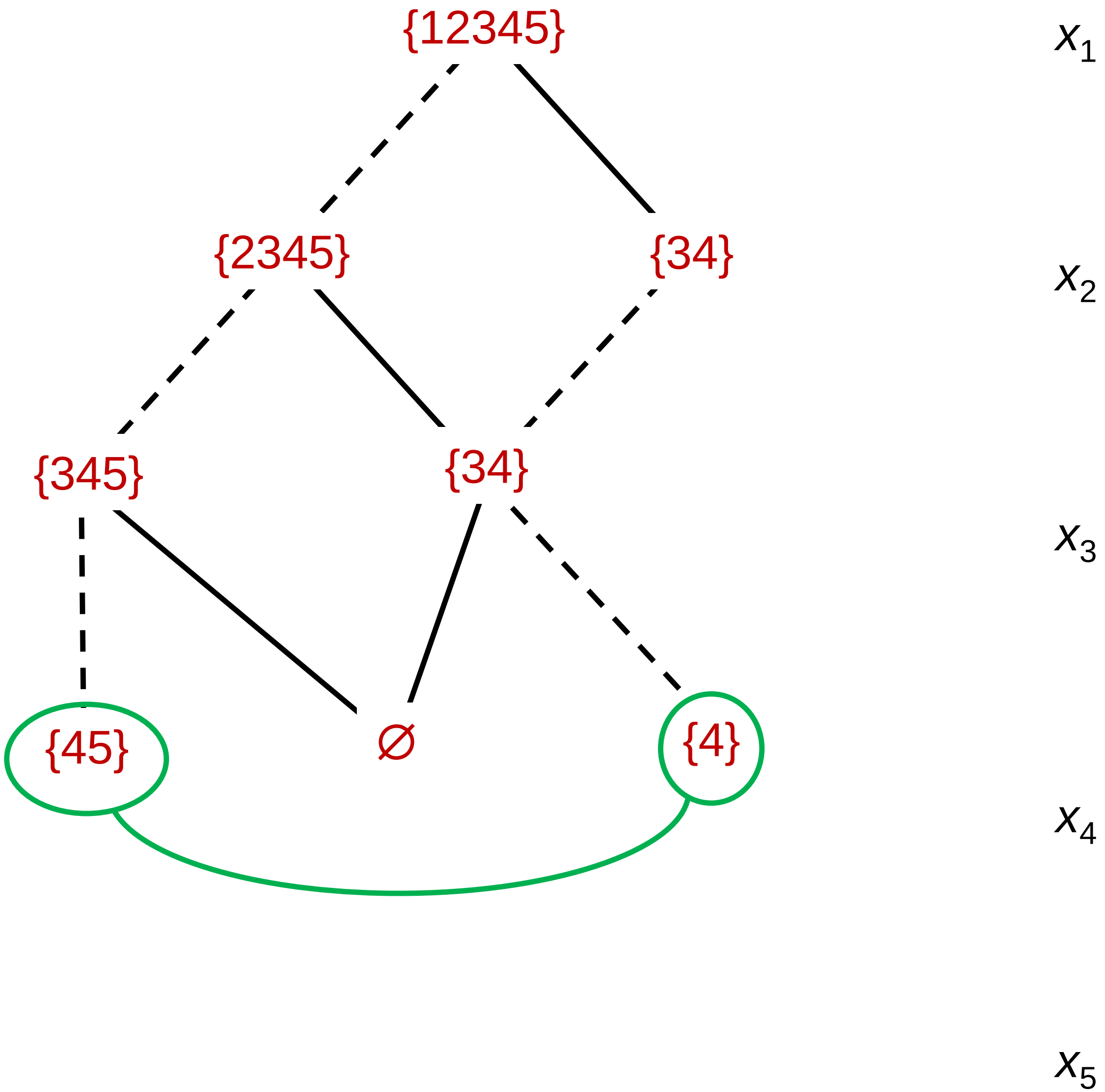

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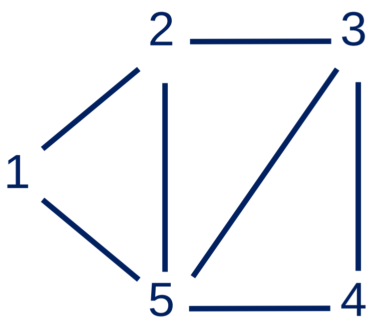

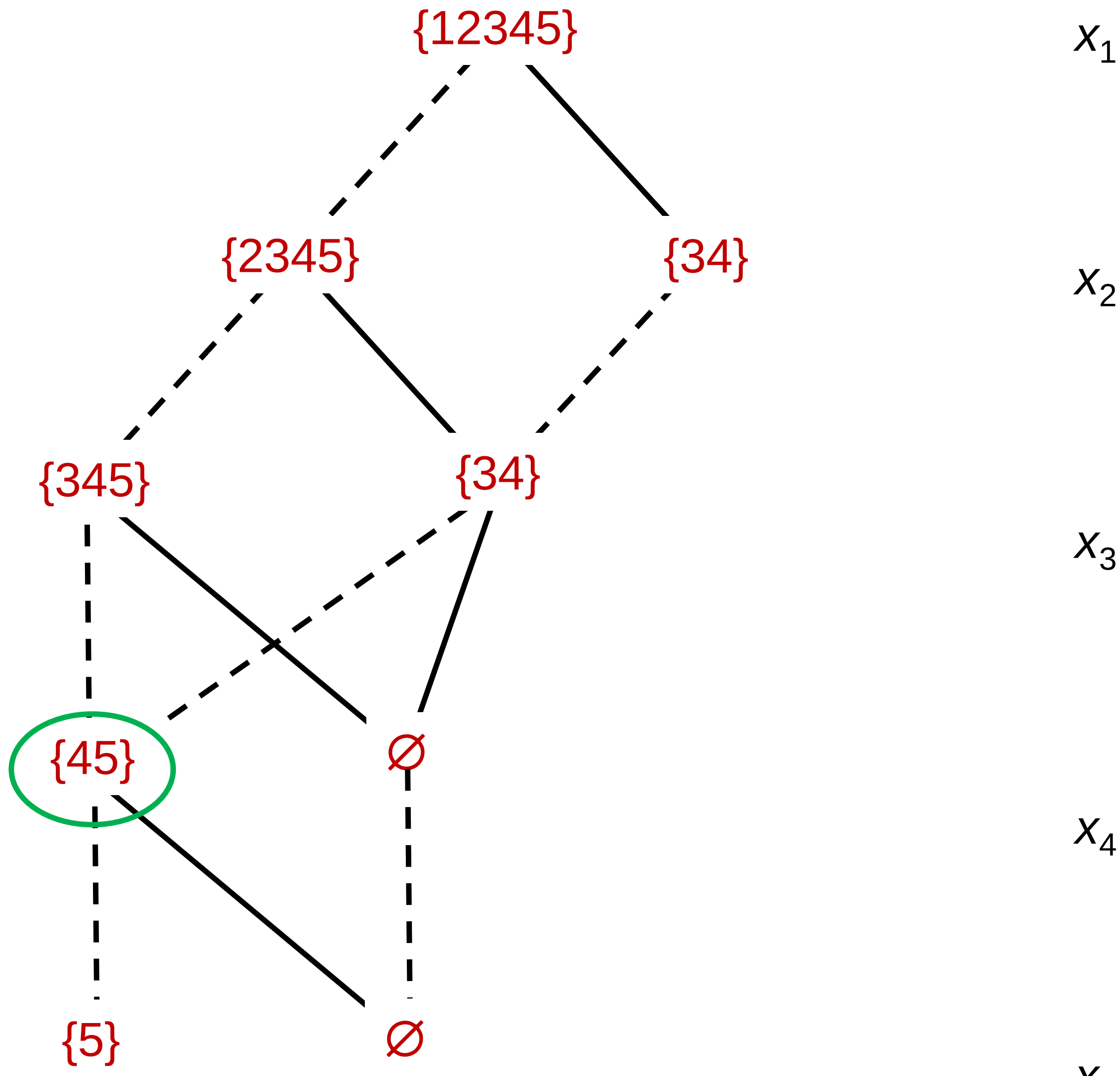

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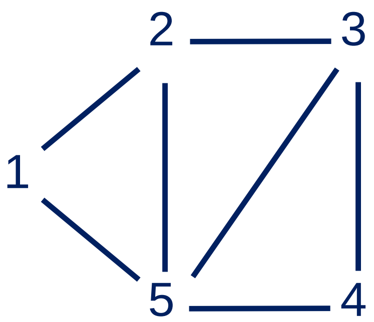
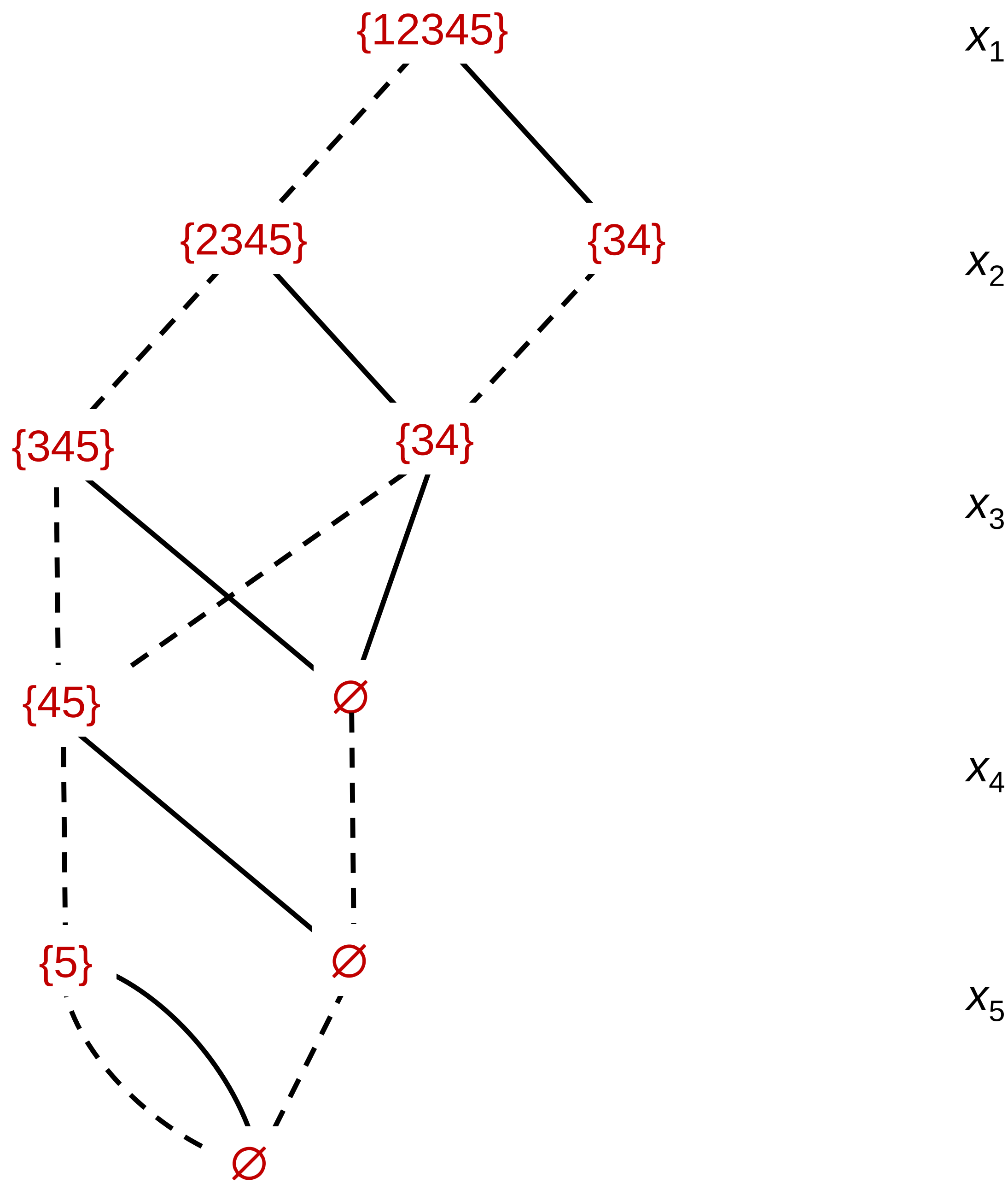
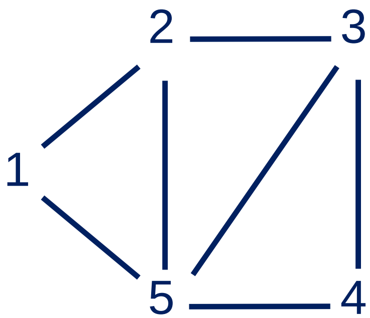


Width = 2

Represents 11 solutions, including 9 feasible solutions





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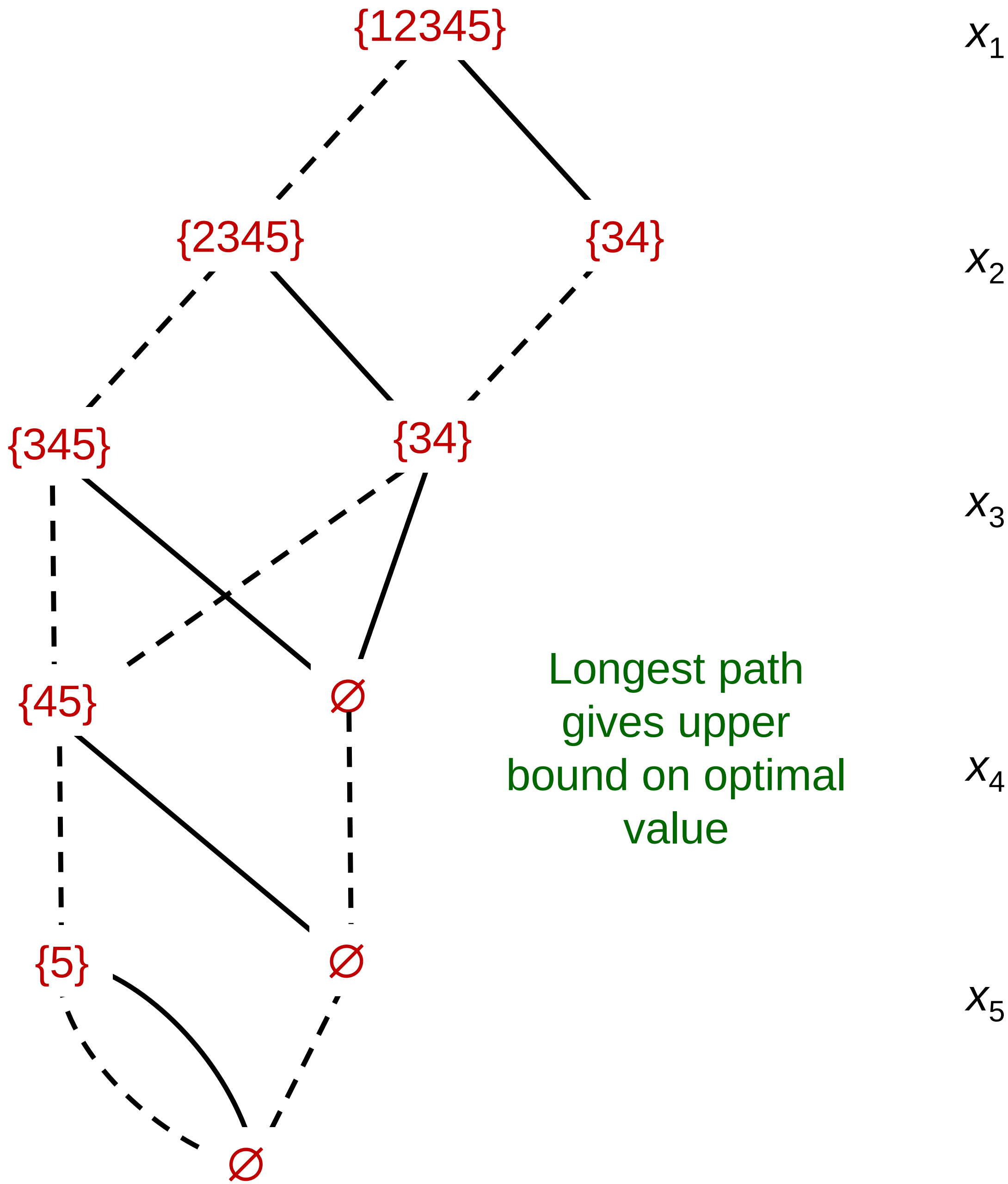
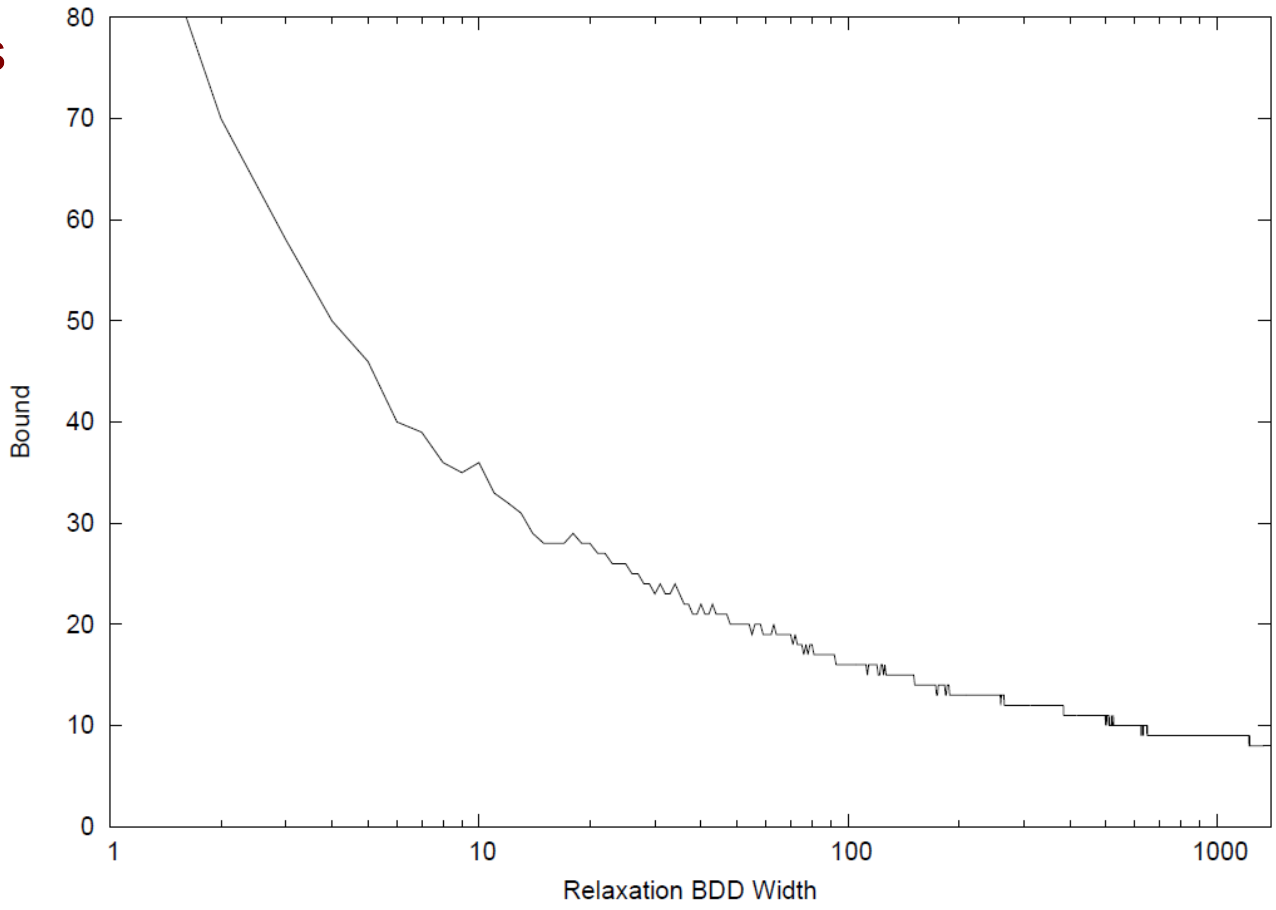


Longest path gives upper bound on optimal value

Decision Diagrams

- Wider diagrams yield tighter bounds
  - But take longer to build.
  - Adjust width dynamically.



Decision Diagrams

- Recursive modeling
  - **Dynamic programming** model of problem, using **state variables**.
    - Using state variables.
  - Rule for **merging states** to create relaxed DD.
    - Analogous to adding valid inequalities in IP.
  - How about **curse of dimensionality**?
    - Solve by **branch and bound** in the relaxed diagram, rather than state space enumeration.

Decision Diagrams

- A novel **branch-and-bound** algorithm.
  - Branch on nodes in **last exact layer** of relaxed decision diagram.
    - …rather than branch on variables.
    - Create a new **relaxed DD rooted** at each branching node.
    - Prune search tree using bounds from relaxed DD.

Decision Diagrams

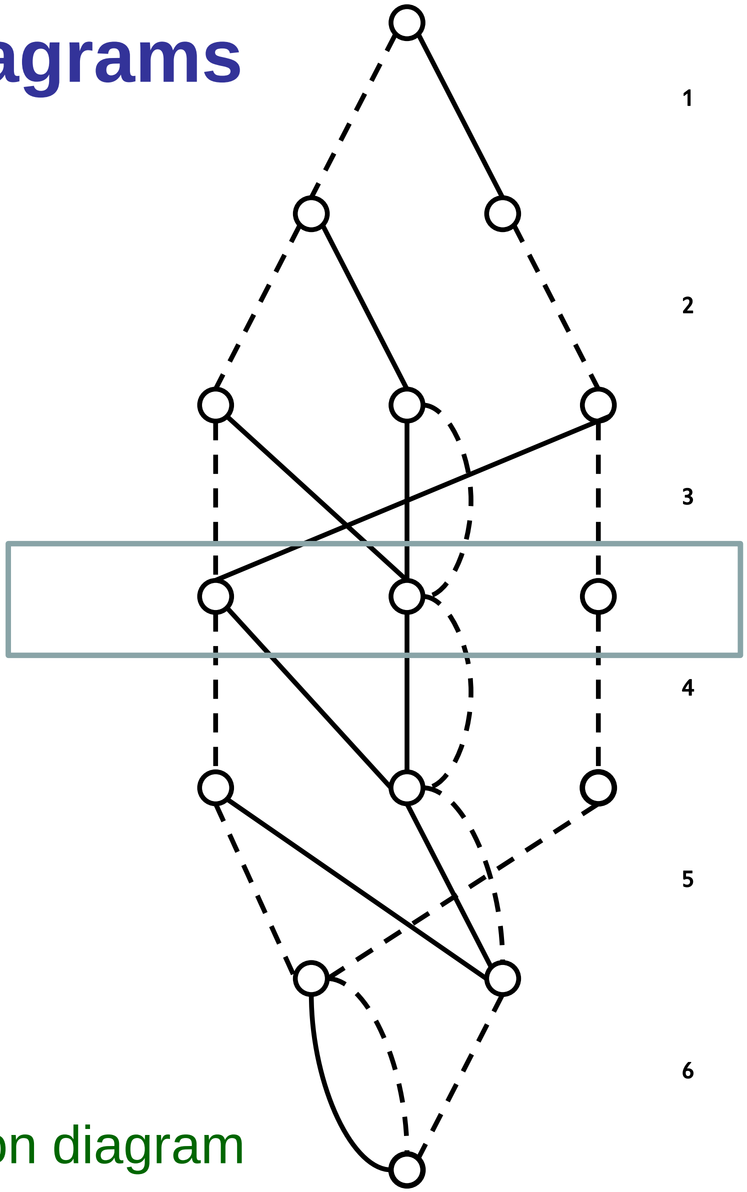
- A novel **branch-and-bound** algorithm.
  - Branch on nodes in **last exact layer** of relaxed decision diagram.
    - …rather than branch on variables.
    - Create a new **relaxed DD rooted** at each branching node.
    - Prune search tree using bounds from relaxed DD.
  - Advantage: a manageable number states may be reachable in first few layers.
    - …even if the state space is **exponential**.

Decision Diagrams

Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

Diagram is exact down to here



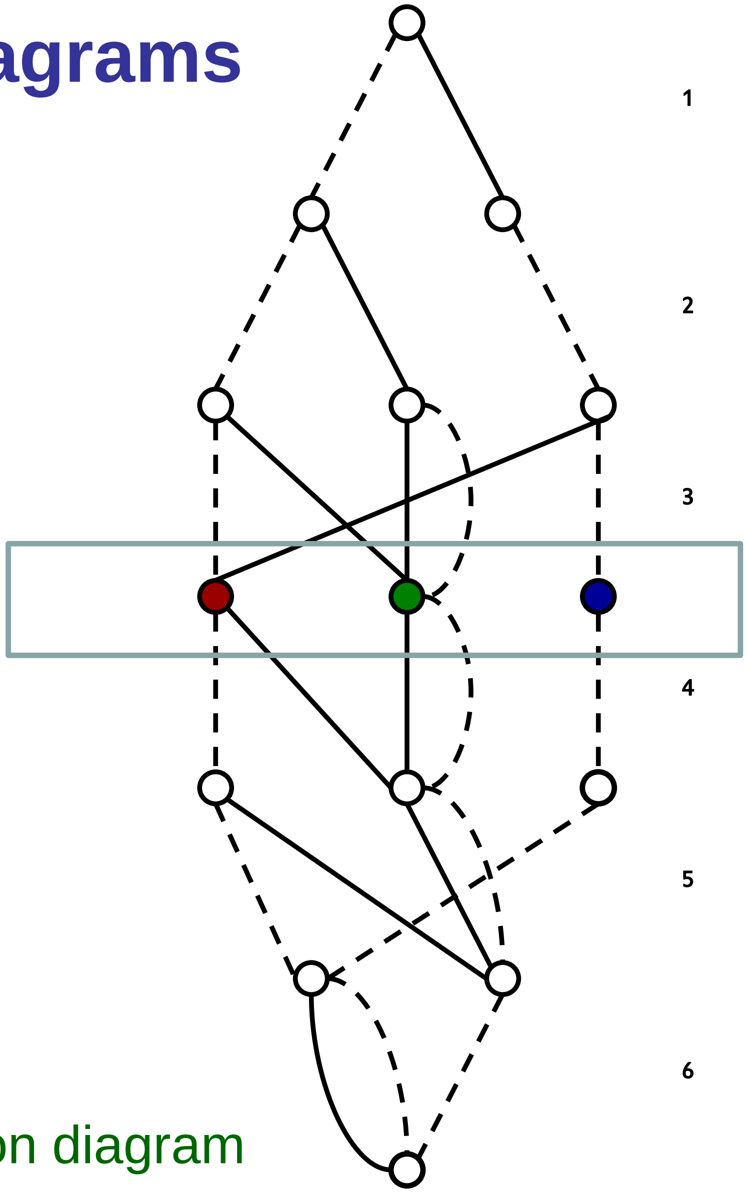
Relaxed decision diagram

Decision Diagrams

Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

Branch on nodes in this layer



Relaxed decision diagram

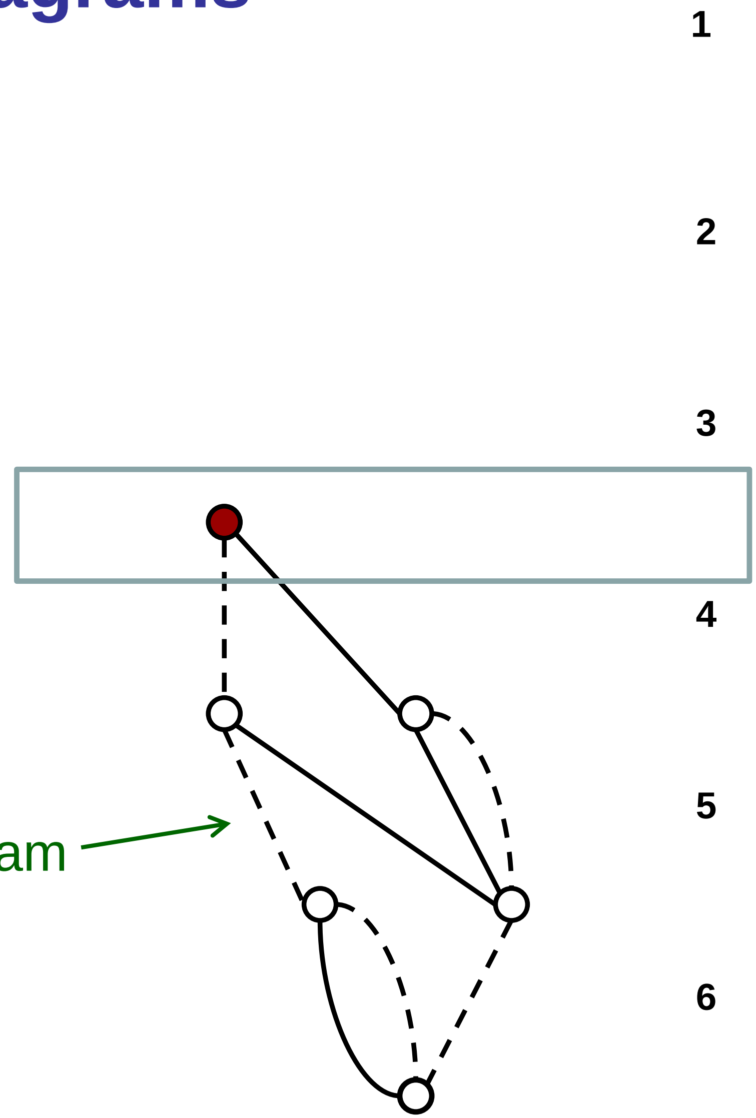
Decision Diagrams

Branching in a relaxed
decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

First branch

New relaxed decision diagram

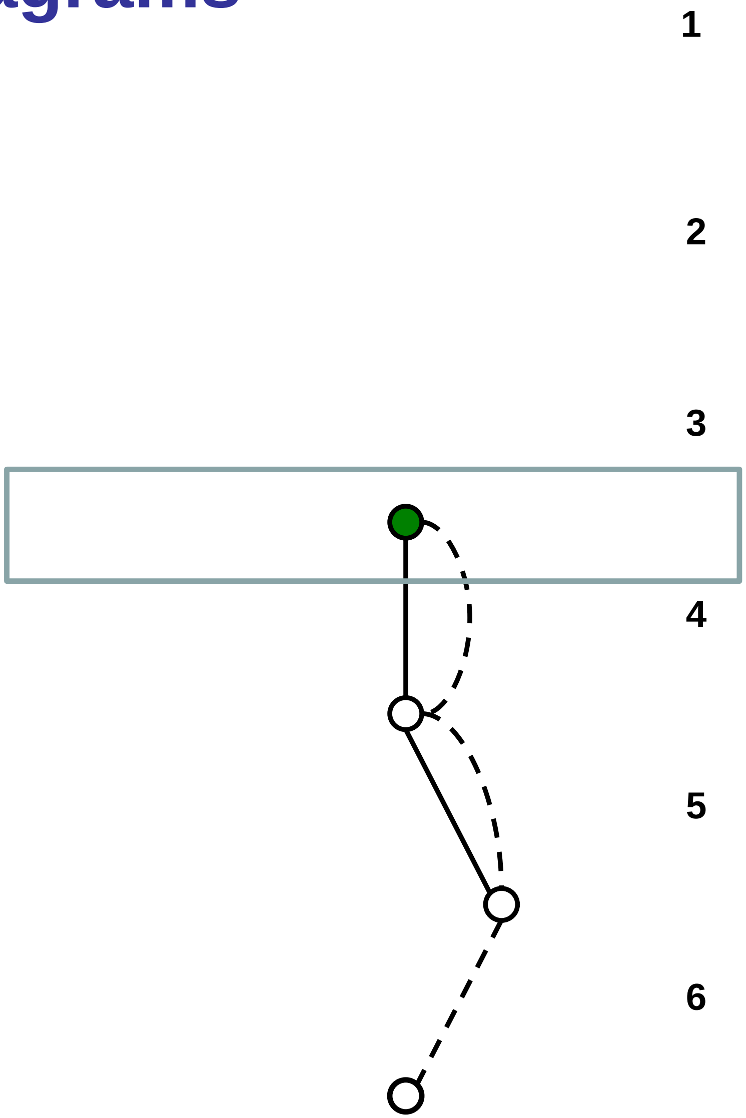


Decision Diagrams

Branching in a relaxed decision diagram

Bergman, Ciré, van Hoeve, JH (2014)

Second branch



Decision Diagrams

Branching in a relaxed
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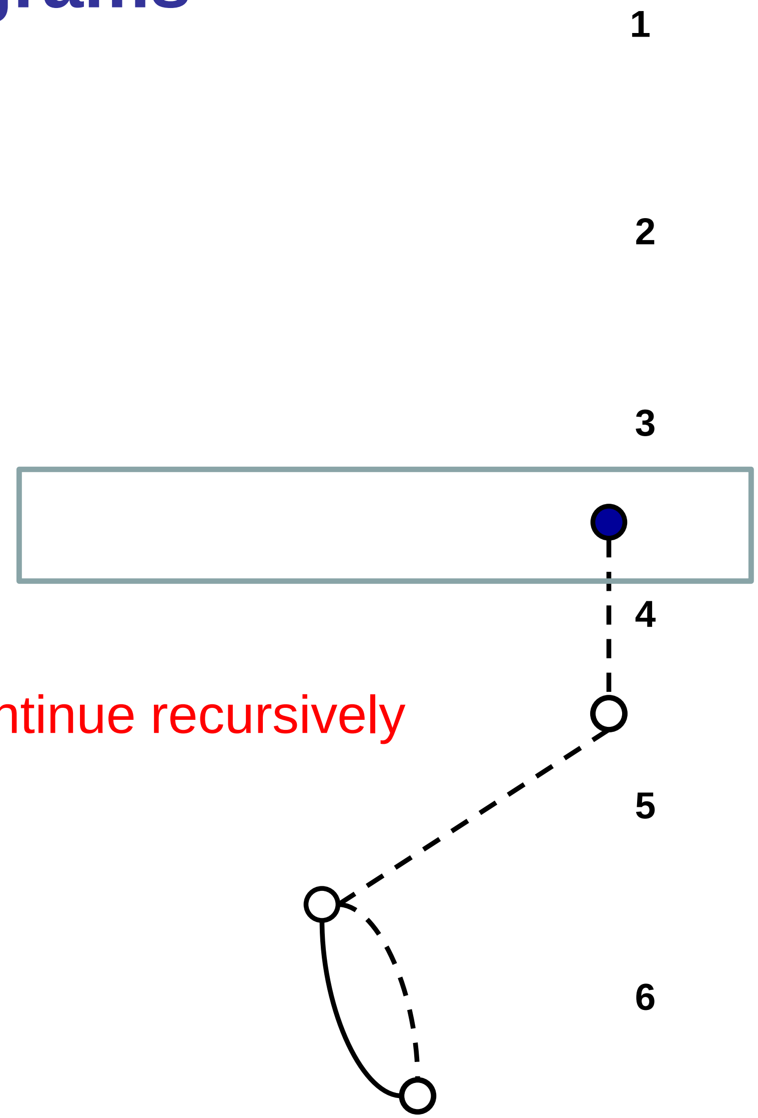
Third branch



Decision Diagrams

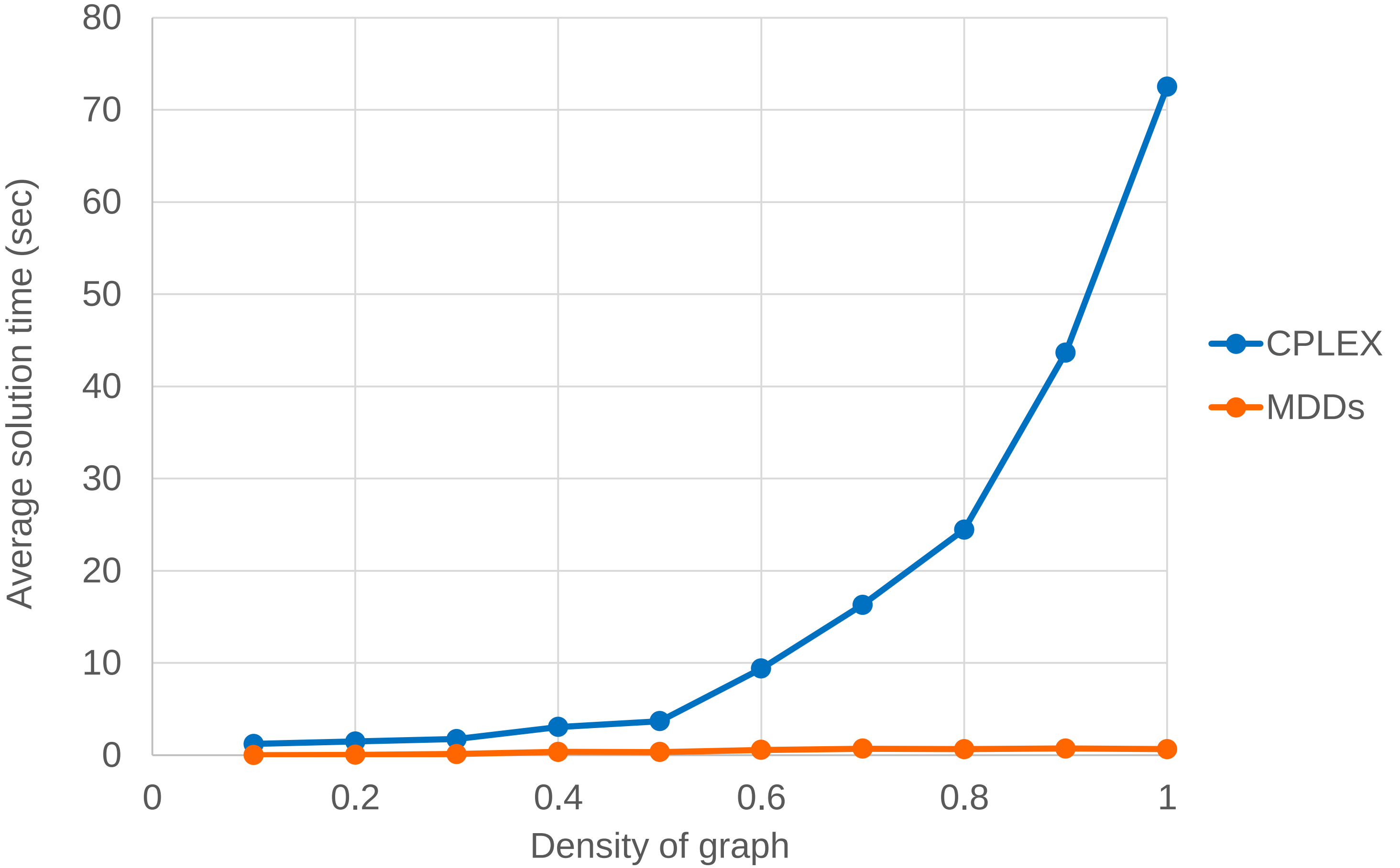
- Computational results for optimization.
  - Stable set, max cut, max 2-SAT.
    - Superior to commercial MIP solver (CPLEX) on most instances.
    - Even though the problems have **natural MIP models**.
    - Obtained best known solution on some max cut instances.
  - Slightly slower than MIP on stable set with precomputed clique cover model, but…

Decision Diagrams

Max cut
on a graph

Avg. solution time
vs
graph density

30 vertices

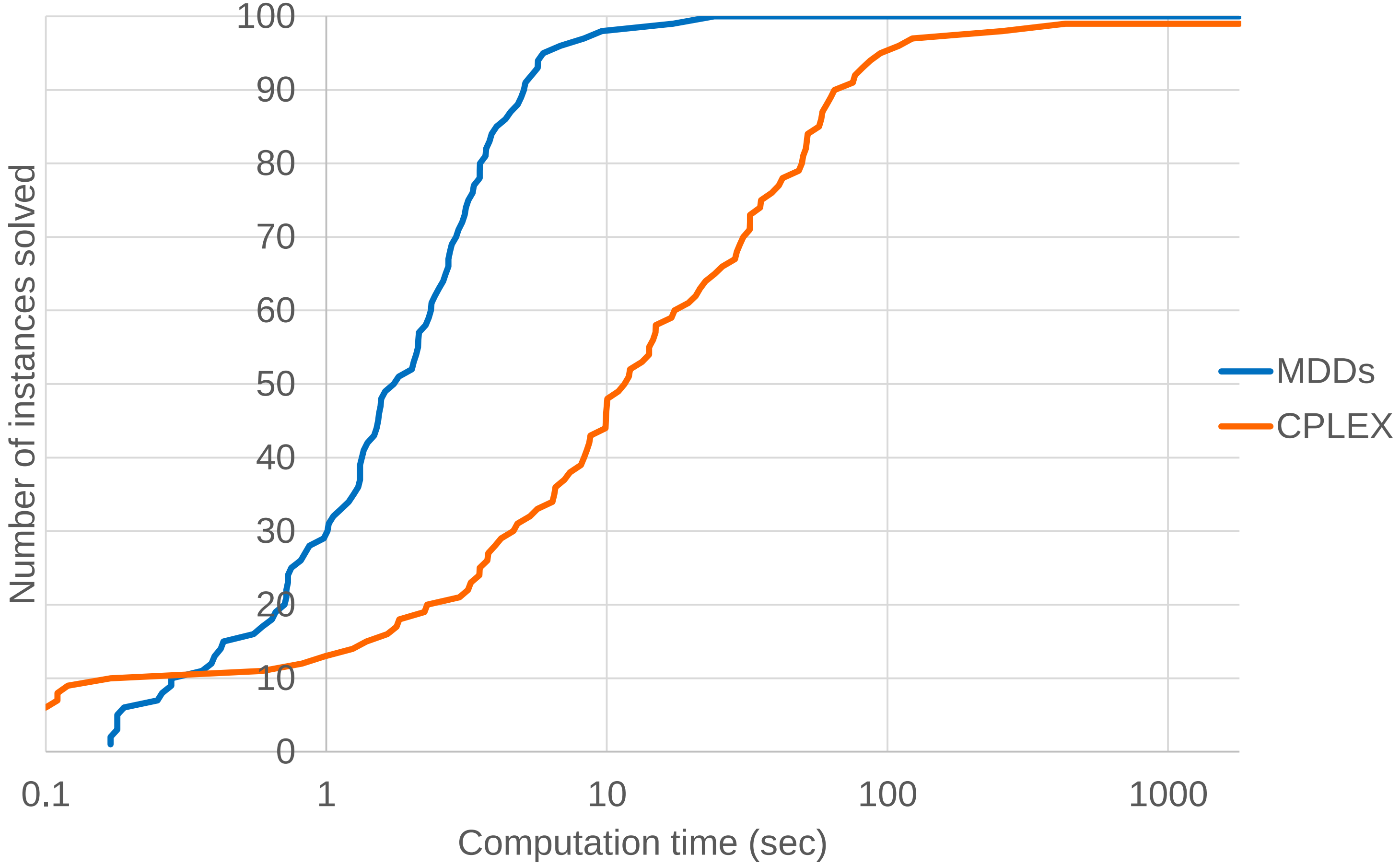


Decision Diagrams

Max 2-SAT

Performance profile

30 variables

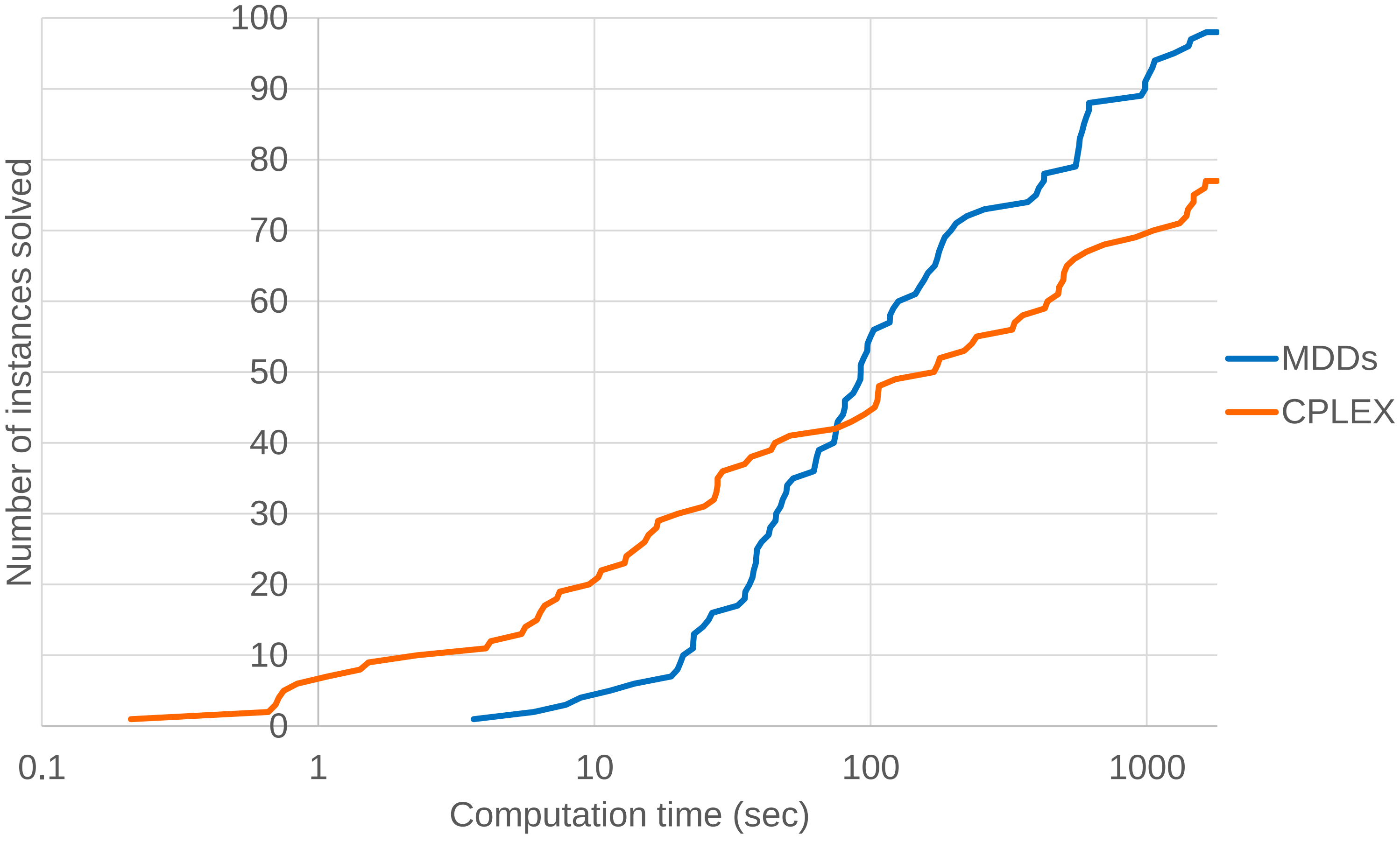


Decision Diagrams

Max 2-SAT

Performance profile

40 variables



Decision Diagrams

- Potential to scale up
  - No need to load large inequality model into solver.
  - Parallelizes very effectively
    - Near-linear speedup.
    - Much better than mixed integer programming.

Decision Diagrams

- Next steps
  - Decision diagram technology is ready for **large-scale** applications from industry
    - …which will also help develop the technology.

Decision Diagrams

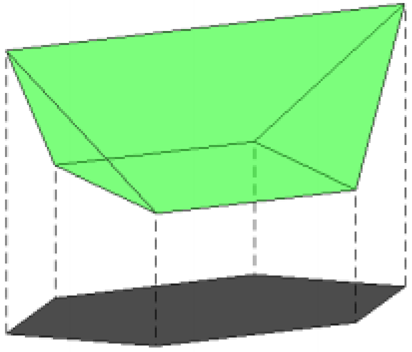
- Next steps
  - Decision diagram technology is ready for **large-scale** applications from industry
    - …which will also help develop the technology.
  - Give problems a **dynamic programming** model.
    - …natural for many applications.
    - But solve by **branch and bound**, not DP.
  - Extend to **stochastic** DP and optimal control.
    - Current research.

Logic-Based Benders

- **Logic-based Benders decomposition** is a generalization of classical Benders decomposition.

JH (1995) | JH and Yan (1995) | JH and Ottosson (2003)

- Solves a **projection** problem.
  - ...and therefore a **logical inference** problem.



Logic-Based Benders

- Collaborators…
  - André Ciré
  - Elvin Çoban
  - Aliza Heching
  - Greger Ottosson
  - Erlendur Thorsteinsson*
  - Hong Yan

***2001** *Best Paper Award*, CP conference

Logic-Based Benders

- Subproblem is an **arbitrary optimization** problem.
  - In classical Benders, subproblem must be linear or nonlinear programming problem.
    - LBBD replaces LP dual with **inference dual**.
  - Solves problems of the form

$$\begin{aligned}&\min f(x,y)\\&(x,y)\in S\\&x\in D\end{aligned}$$

Logic-Based Benders

- Decompose problem into master and subproblem.
  - Subproblem is obtained by fixing x to solution value in master problem.

Master problem

$$\min z$$
$$z \geq g_k(x) \quad \text{(Benders cuts)}$$
$$x \in D$$

Find x that minimizes cost z subject to Benders cuts obtained from solutions in previous iterations k.

Trial value $\bar{x}$ that solves master

Benders cut $z \geq g_k(x)$

Subproblem

$$\min f(\bar{x}, y)$$
$$(\bar{x}, y) \in S$$

Obtain proof of optimality (**inference dual**).
Use same proof to deduce cost bounds for other assignments, yielding **Benders cut**.

Logic-Based Benders

- Iterate until master problem value equals best subproblem value so far.
  - This yields optimal solution.

Master problem

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$$z \ge g_k(x) \quad \text{(Benders cuts)}$$
$$x \in D$$

Find x that minimizes cost z subject to Benders cuts obtained from solutions in previous iterations k.

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- Some applications:
  - Circuit verification
  - Chemical batch processing (BASF, etc.)
  - Steel production scheduling
  - Auto assembly line management (Peugeot-Citroën)
  - Automated guided vehicles in flexible manufacturing
  - Allocation and scheduling of multicore processors (IBM, Toshiba, Sony)
  - Facility location-allocation
  - Stochastic facility location and fleet management
  - Capacity and distance-constrained plant location

Logic-Based Benders

- Some applications…
  - Transportation network design
  - Traffic diversion around blocked routes
  - Worker assignment in a queuing environment
  - Single- and multiple-machine allocation and scheduling
  - Permutation flow shop scheduling with time lags
  - Resource-constrained scheduling
  - Wireless local area network design
  - Service restoration in a network
  - Optimal control of dynamical systems
  - Sports scheduling

Logic-Based Benders

- Application to planning and scheduling.
  - Assign tasks in master, schedule them in subproblem.
    - Combine **mixed integer programming (MIP)** and **CP**

Master problem

Assign tasks to resources to minimize cost.

Solve by **MIP**.

Trial assignment $\bar{x}$

Benders cut $z \geq g_k(x)$

Subproblem

Schedule tasks on each resource, subject to time windows.

CP obtains proof of optimality (inference dual).

Use **same proof** to deduce cost for other assignments, yielding **Benders cut**.

Logic-Based Benders

- Objective function
  - Cost is based on **task assignment only**.

$$\text{cost } = \sum_{ij} c_{ij} x_{ij}, \quad x_{ij} = 1 \text{ if task } j \text{ assigned to resource } i$$

  - So cost appears only in the **master problem**.
  - Scheduling subproblem is a **feasibility problem**.

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  - Cost is based on **task assignment only**.

    $$\text{cost} = \sum_{ij} c_{ij} x_{ij}, \quad x_{ij} = 1 \text{ if task } j \text{ assigned to resource } i$$

    - So cost appears only in the **master problem**.
    - Scheduling subproblem is a **feasibility problem**.

- Benders cuts
  - They have the form $\sum_{j \in J_i} (1 - x_{ij}) \geq 1, \text{ all } i$
    - where J_i is a set of tasks that create infeasibility when assigned to resource i.

Logic-Based Benders

- Resulting Benders decomposition:

Master problem

$$\min\ z$$
$$z = \sum_{ij} c_{ij} x_{ij}$$
Benders cuts

Trial assignment $\bar{x}$

Benders cuts

$$\sum_{j \in J_i} (1 - x_{ij}) \geq 1,$$

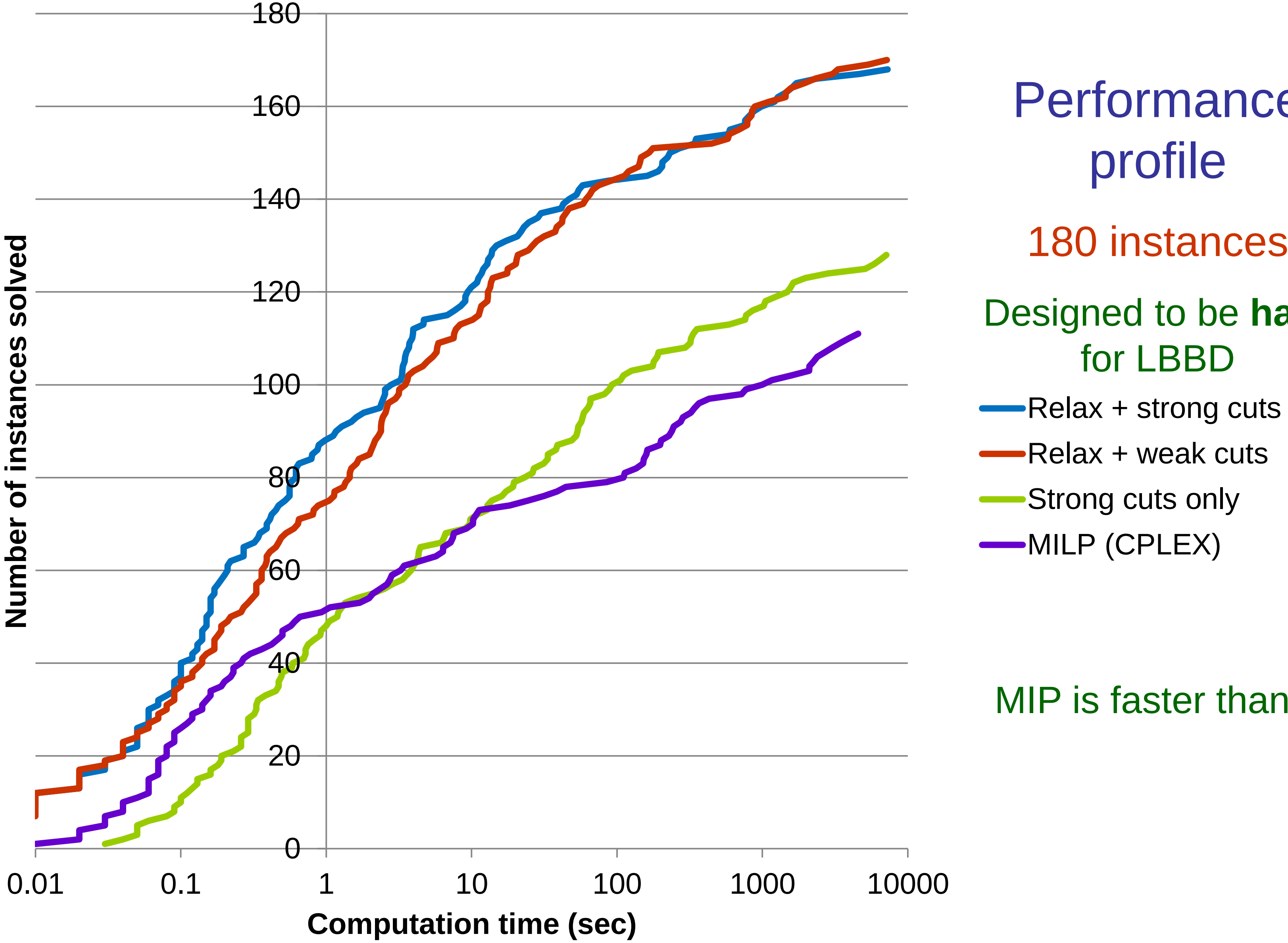
for infeasible resources *i*

Subproblem

Schedule jobs on each resource.

CP may obtain proof of infeasibility on some resources (inference dual).

Use **same proof** to deduce infeasibility for some other assignments, yielding **Benders cut**.



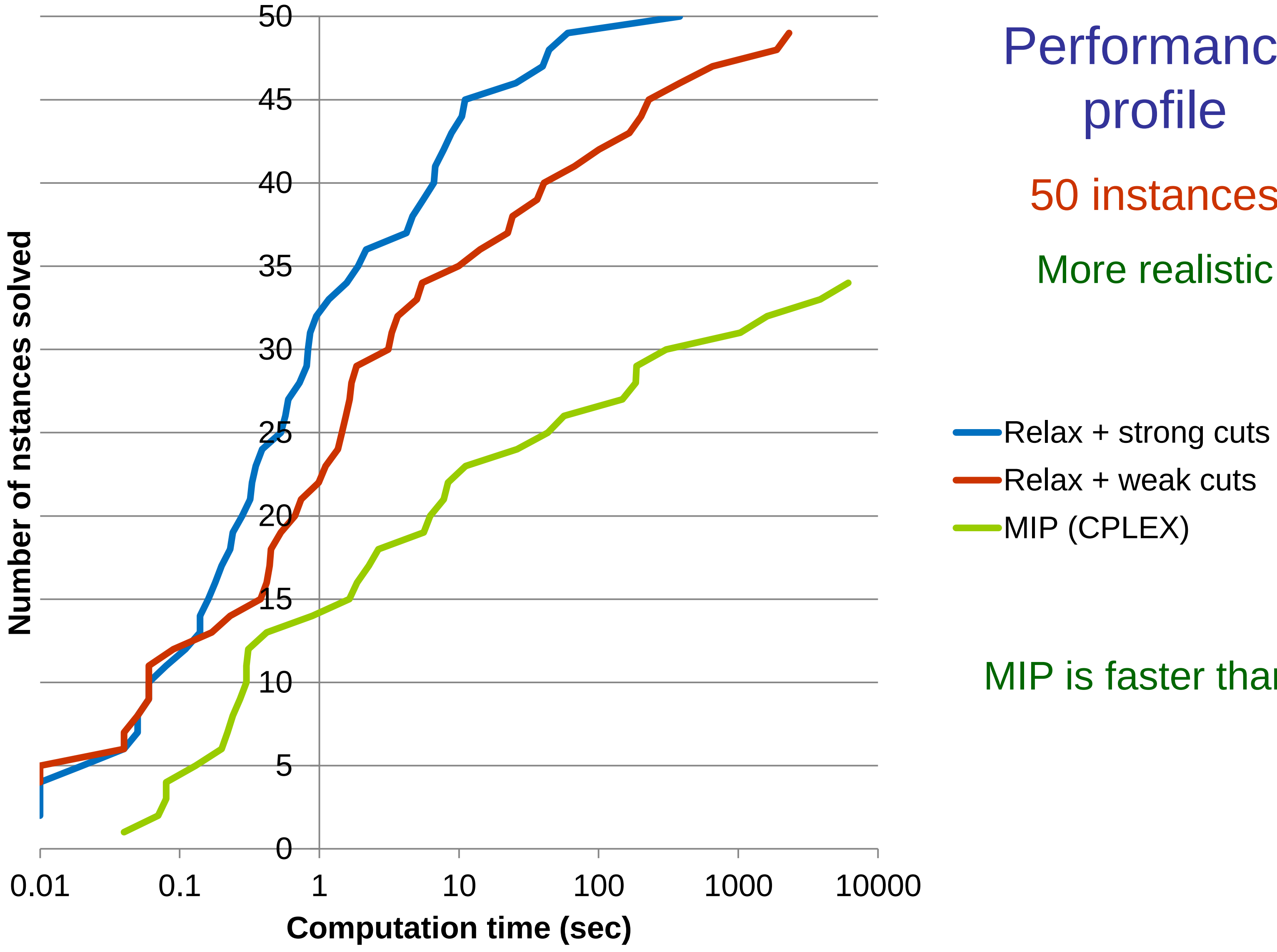
Performance profile

180 instances

Designed to be **hard** for LBBD

MIP is faster than CP

Ciré, Çoban, JH (2015)



Performance profile

50 instances

More realistic

MIP is faster than CP

Ciré, Çoban, JH (2015)

Logic-Based Benders

- Current research
  - Apply to **robust** scheduling.
  - Use a **decision diagram** in the master problem.
  - Apply to **logical inference** from large datasets.

Logic-Based Benders

- Robust scheduling
  - …with uncertainty sets.
    - Uncertainty subproblem becomes Benders subproblem.
  - IT service center scheduling.
    - Projects subject to uncertain delays
    - Give customers a reasonable worst-case completion date.

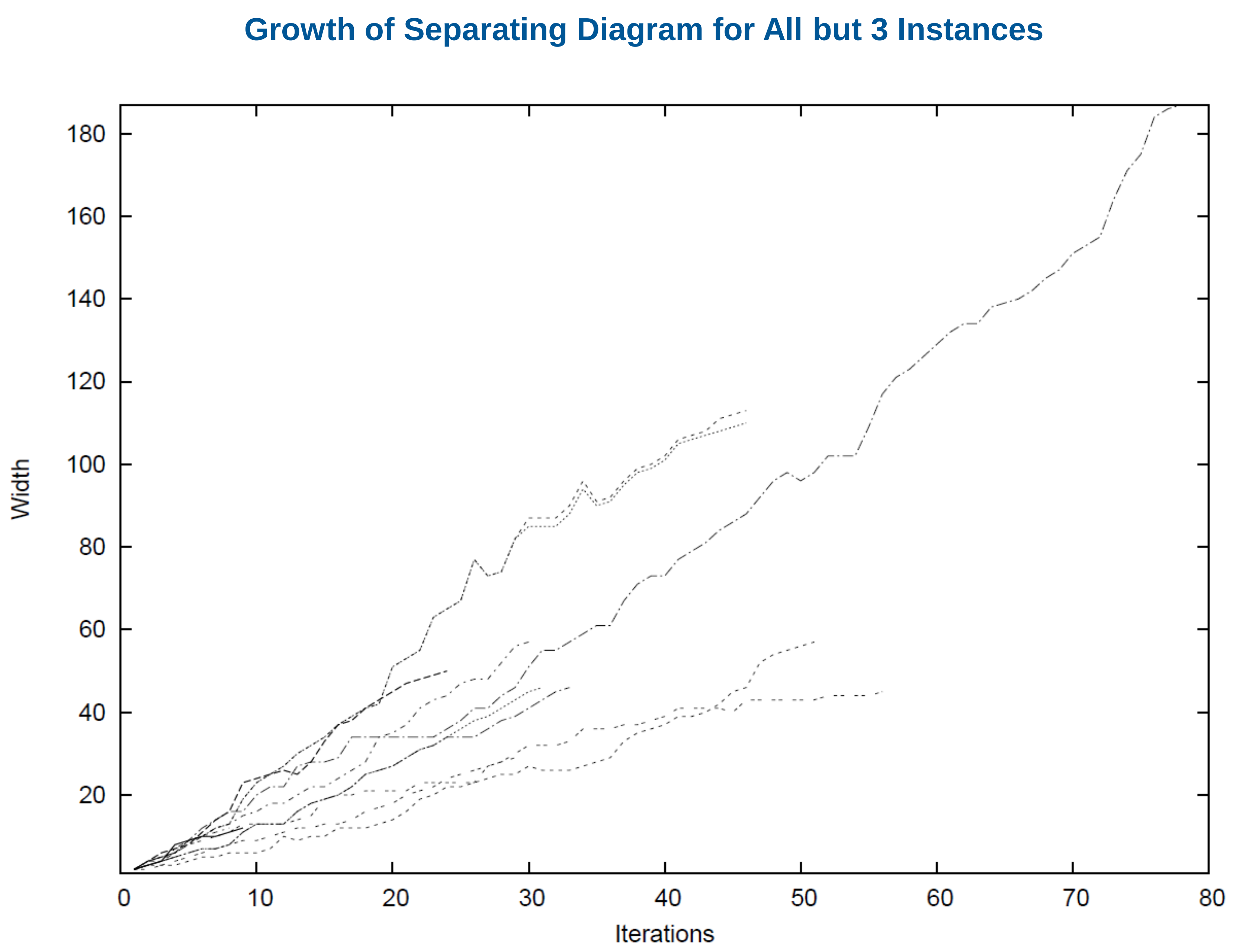
Çoban, Heching, JH (2015)

Logic-Based Benders

- **Decision diagram** in master problem.
  - Add Benders cuts by **modifying** the decision diagram.
    - Requires **separation algorithm** for decision diagrams
  - Home healthcare scheduling.
    - Master problem assigns healthcare aides to patients.
    - Subproblem schedules visits and routes the aides.
  - Key question:
    - How much will the decision diagram **grow** as Benders cuts are added?

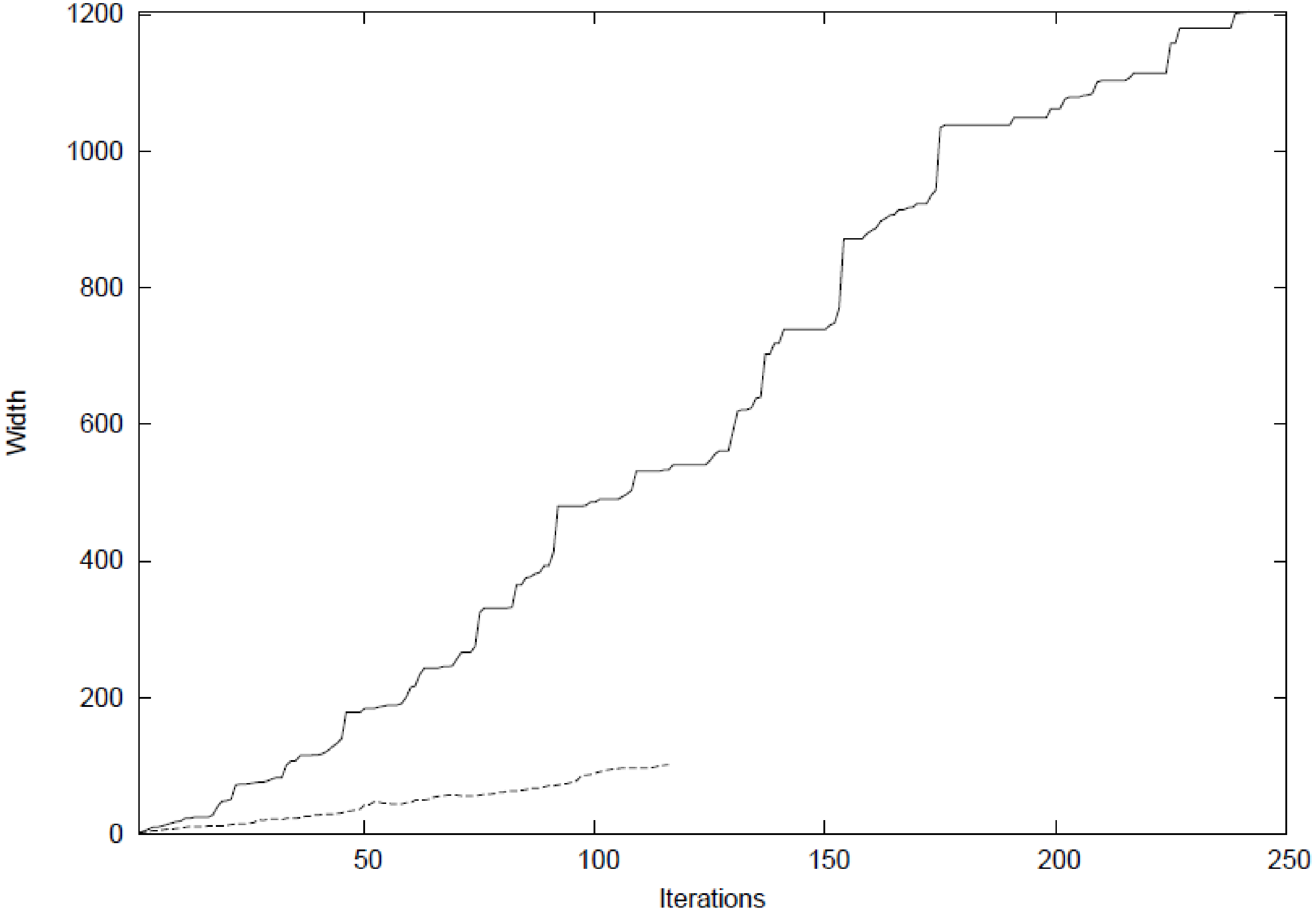
Ciré, JH (2014)

Heching, JH (2015)

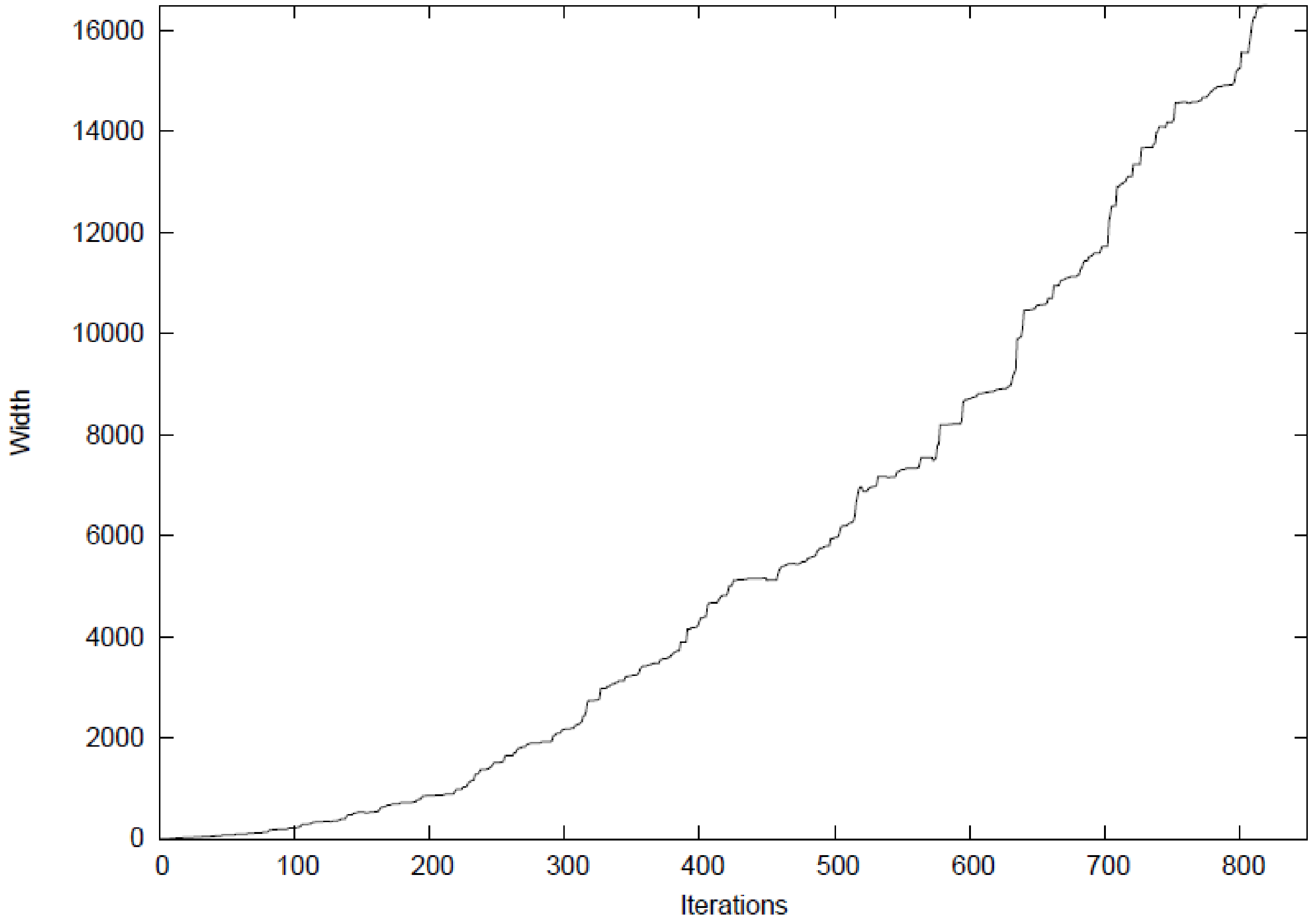


Growth of Separating Diagram for All but 3 Instances
Width
Iterations
0
10
20
30
40
50
60
70
80
20
40
60
80
100
120
140
160
180

Growth of Separating Diagram for 2 Harder Instances

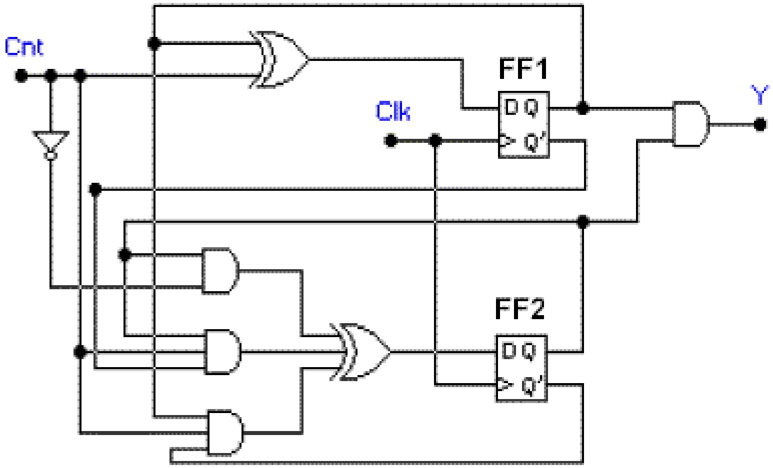


Growth of Separating Diagram for Hardest Instance



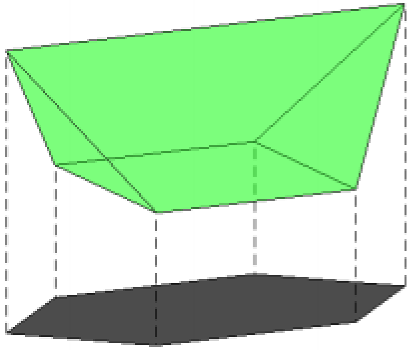
Logic-Based Benders

- Logical inference.
  - Benders cuts describe the **projection** of the feasible set onto the master problem
  - This is the fundamental **logical inference** problem.
    - Original application was to logic circuit verification.



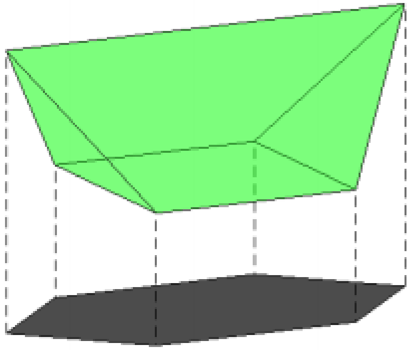
Logical Inference

- Connection between optimization and logic.
  - Both are **projection** problems.
  - Projection = extract information involving a subset of variables.



Logical Inference

- Connection between optimization and logic.
  - Both are **projection** problems.
  - Projection = extract information involving a subset of variables.
- Optimization…
  - **Project** feasible set onto a single variable representing the objective function.
- Inference…
  - **Project** knowledge base onto a subset of propositional variables.



Logical Inference

- Example:
  - Knowledge base consists of logical clauses.
  - Derive all implications involving x_1, x_2.
  - This is a projection problem.

$$x_1 \vee x_2$$
$$\neg x_1 \vee x_3$$
$$\neg x_1 \vee \neg x_2 \vee \neg x_3$$
$$x_1 \vee x_3 \vee x_4$$
$$x_2 \vee x_3 \vee \neg x_4$$

$$x_1 \vee x_2$$
$$\neg x_1 \vee \neg x_2$$

Projection

Logical Inference

- Example:
  - Knowledge base consists of logical clauses.
  - Derive all implications involving x_1, x_2.
  - This is a projection problem.
- Solved by logic-based Benders.
  - Clauses in the projection are Benders cuts
    - They are also **conflict clauses** containing x_1, x_2 from a SAT algorithm.

$$\boxed{\begin{array}{l} x_1 \vee x_2 \\ \neg x_1 \vee x_3 \\ \neg x_1 \vee \neg x_2 \vee \neg x_3 \\ x_1 \vee x_3 \vee x_4 \\ x_2 \vee x_3 \vee \neg x_4 \end{array}}$$

$$\boxed{\begin{array}{l} x_1 \vee x_2 \\ \neg x_1 \vee \neg x_2 \end{array}}$$

Projection

Logical Inference

- A central problem today:
  - Associate each inference with its **probability**, **relevance**, or **confidence**.
    - For example, IBM's **Watson** (Jeopardy player).

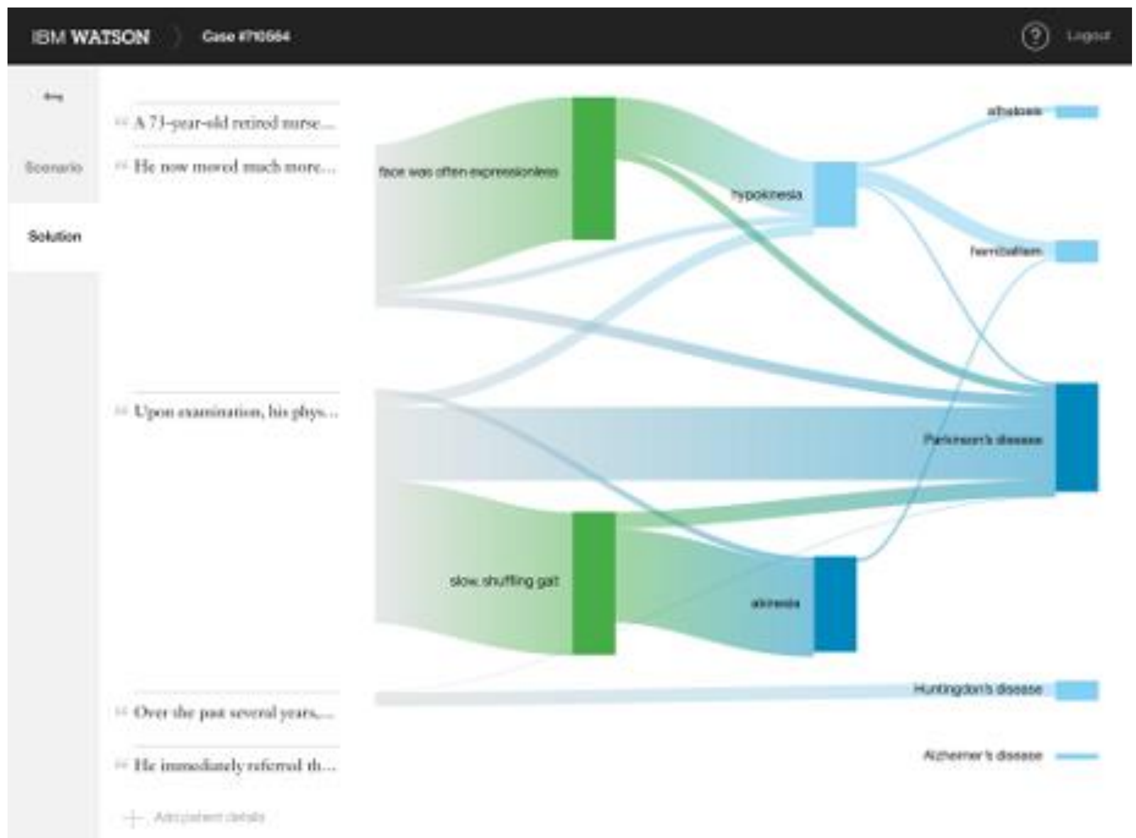
Logical Inference

Watson technology first applied to medicine (WatsonPaths).

Draws inferences from medical literature and clinical guidelines.

About 1 million articles listed per year in PubMed.

Probably 1.5-2 million overall.

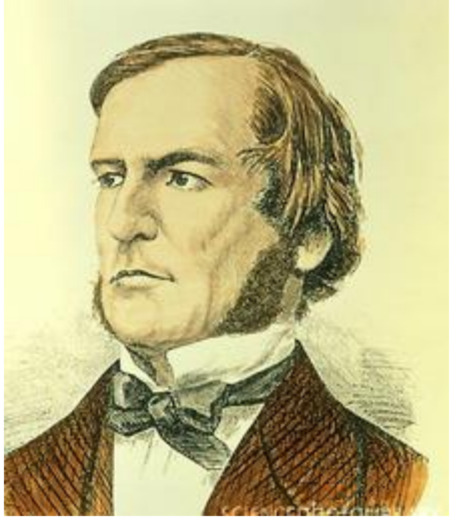


Logical Inference

- The problem was first posed for **probability**...

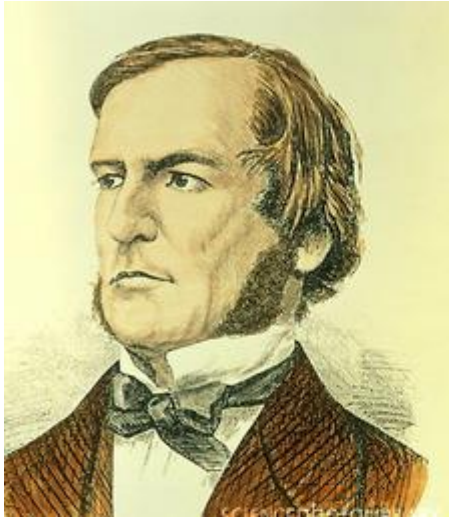
Logical Inference

- The problem was first posed for **probability**...
  - …by George Boole.
    - He viewed **probability logic** as a key contribution.



Logical Inference

- The problem was first posed for **probability**...
  - …by George Boole.
    - He viewed **probability logic** as his most important contribution.
    - Formulated inference problem as **linear programming**.
    - Can be solved by modern **column generation** methods.



Logical Inference

Example

Clause	Probability
x_1	0.9
$\bar{x}_1 \vee x_2$	0.8
$\bar{x}_2 \vee x_3$	0.4

Deduce probability range for x_3

Logical Inference

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Deduce probability range for x_3

Linear programming model

$$\min/\max \ \pi_0$$

$$\begin{bmatrix} 01010101 \\ 00001111 \\ 11110011 \\ 11011101 \\ 11111111 \end{bmatrix} \begin{bmatrix} p_{000} \\ p_{001} \\ p_{010} \\ \vdots \\ p_{111} \end{bmatrix} = \begin{bmatrix} \pi_0 \\ 0.9 \\ 0.8 \\ 0.4 \\ 1 \end{bmatrix}$$

p_{000} = probability that $(x_1,x_2,x_3) = (0,0,0)$

Logical Inference

Example

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p_{000} = probability that $(x_1,x_2,x_3) = (0,0,0)$

Solution: $\pi_0 \in [0.1,0.4]$

Logical Inference

- Next steps
  - Apply **large-scale optimization** methods to logical inference.
    - They are substantially **underutilized** for this purpose.
  - In particular:
    - Use **logic-based Benders** for propositional logic, etc.
    - Use **column generation** for probability logic.
    - Use **linear programming** for belief logics, relevance, and variations of Dempster-Shafer theory.
    - Use **nonlinear programming** for Bayesian networks.
    - Use **decision diagrams** for queries and what-if analysis.

Andersen, JH (1992,1994,1996)

Chandru, JH (1999)