# Constraint Programming Techniques in MINLP

J. N. Hooker

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# What can CP contribute to MINLP?

- **Interval propagation** (range reduction).
  - Already used in global optimization solvers.
- Focus on lesser-known techniques:
  - **Domain filtering** with Lagrange multipliers.
  - Efficient representation of **piecewise linear** functions.
  - Branching on **multiple discrete values**.
  - Bounds from **quasi-relaxations**.
  - Optimization/propagation in **decision diagrams**.
  - Management of **McCormick factors** with global constraints and semantic typing (*no time for this today*).

# Constraint Programming Perspective

- All (successful) optimization method combine **search** with **relaxation** and **inference**.
- Math programming focuses on **relaxation**.
  - LP, Lagrangean, etc.
- Constraint programming (CP) focuses on **inference**.
  - Domain filtering, constraint propagation

# Constraint Programming Perspective

- All (successful) optimization method combine **search** with **relaxation** and **inference**.
- Math programming focuses on **relaxation**.
  - LP, Lagrangean, etc.
- Constraint programming (CP) focuses on **inference**.
  - Domain filtering, constraint propagation
- Math programming uses inference…
  - To generate cutting planes, Benders cuts, etc.
  - But the purpose is to strengthen the **relaxation**.

# Constraint Programming Perspective

- CP uses inference for **consistency maintenance**
  - …rather than to strengthen a relaxation.
- Greater consistency means **less backtracking** during search.
  - The concept of consistency never developed in math programming, but it helps to explain search behavior.

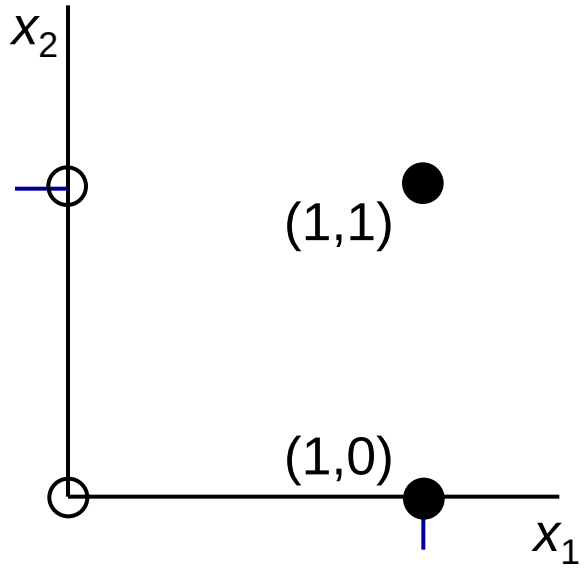
# Constraint Programming Perspective

- CP uses inference for **consistency maintenance**
  - …rather than to strengthen a relaxation.
- Greater consistency means **less backtracking** during search.
  - The concept of consistency never developed in math programming, but it helps to explain search behavior.
- Several types of consistency
  - Domain consistency (generalized arc consistency)
  - Bounds consistency
  - Strong k-consistency
  - Etc. etc.

# Domain consistency

A constraint set is **domain consistent** if the domain of each variable $x_i$ is the projection of the feasible set onto $x_i$.

$$x_1 + x_2 \geq 1$$
$$x_1 - x_2 \geq 0$$
$$x_1, x_2 \in \{0,1\}$$



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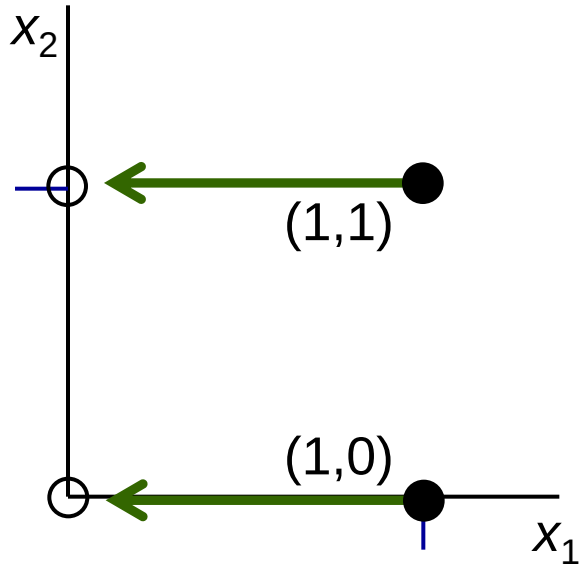


Projection onto $x_2 = \{0,1\}$

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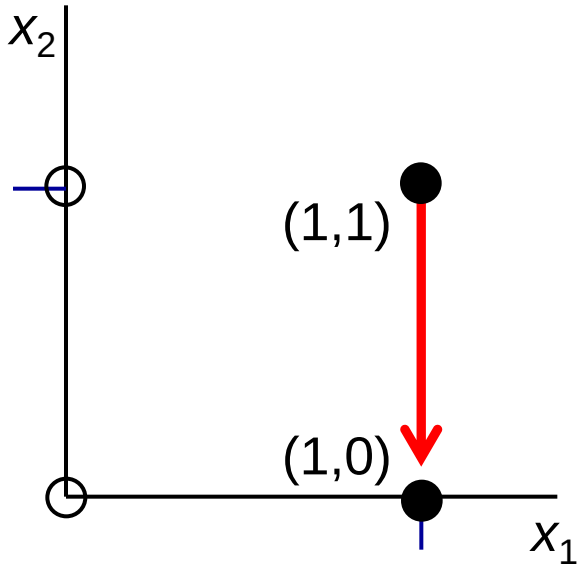



Projection onto $x_2 = \{0,1\}$

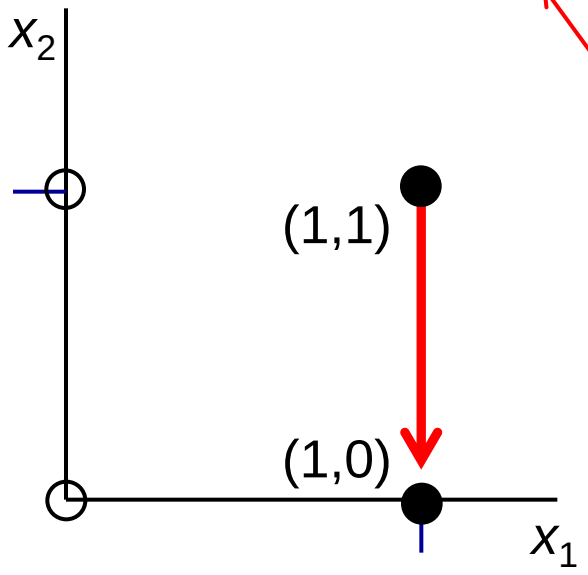
**Not** domain consistent.

Projection onto $x_1 = \{1\}$

# Domain consistency

A constraint set is **domain consistent** if the domain of each variable $x_i$ is the projection of the feasible set onto $x_i$.

$$x_1 + x_2 \geq 1$$
$$x_1 - x_2 \geq 0$$
$$x_1 \in \{1\}, x_2 \in \{0,1\}$$




Achieve domain consistency by **filtering** 0 from domain of $x_1$.

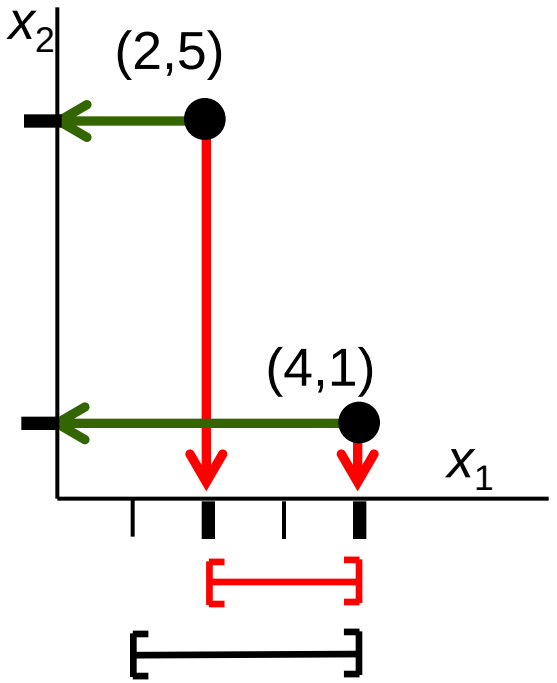
Projection onto $x_2 = \{0,1\}$

Projection onto $x_1 = \{1\}$

# Bounds consistency

$$2x_1 + x_2 = 9$$
$$x_1 \in \{1,2,3,4\}$$
$$x_2 \in \{1,5\}$$

• A constraint set is **bounds consistent** if the domain of each variable $x_i$ spans the **same interval** as the projection of the feasible set onto $x_i$.




**Not** bounds consistent.

# Consistency maintenance

- Domain or bounds consistency is normally achieved (if at all) for **one constraint at a time**.
  - This can be NP-hard.
  - But allows one to exploit **special structure** of constraints.
  - Much as cutting planes exploit structure of certain classes of inequalities.

# Consistency maintenance

- Domain or bounds consistency is normally achieved (if at all) for **one constraint at a time**.
  - This can be NP-hard.
  - But allows one to exploit **special structure** of constraints.
  - Much as cutting planes exploit structure of certain classes of inequalities.
- Particularly effective when the model consists of **global constraints**.
  - …which represent a set of simpler constraints.
  - Alldiff, cardinality, element, nvalues, sequence, circuit, path, regular, cumulative, stretch, etc.

# Propagation

- Reduced domains are passed (**propagated**) to the next constraint.
  - …which may allow further reduction.
  - Generally does not achieve consistency for entire constraint set.
  - But it drastically reduces backtracking.

# Bounds propagation

- Bounds obtained by achieving bound consistency can be propagated.
  - This is important in global optimization (**range reduction**).

$x_1$

# Bounds propagation

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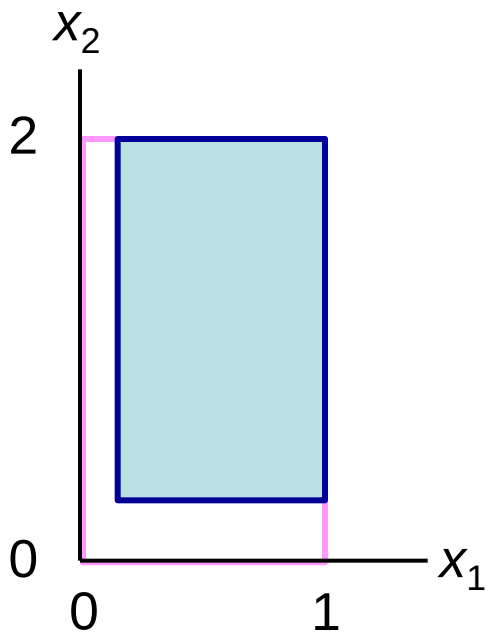
  - This is important in global optimization (**range reduction**).
- Example:

$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0.125, 1]$$
$$x_2 \in [0.25, 2]$$

Filter using constraint 1:

$$x_1 = \frac{1}{4x_2} \ge \frac{1}{4 \cdot 2} = 0.125$$

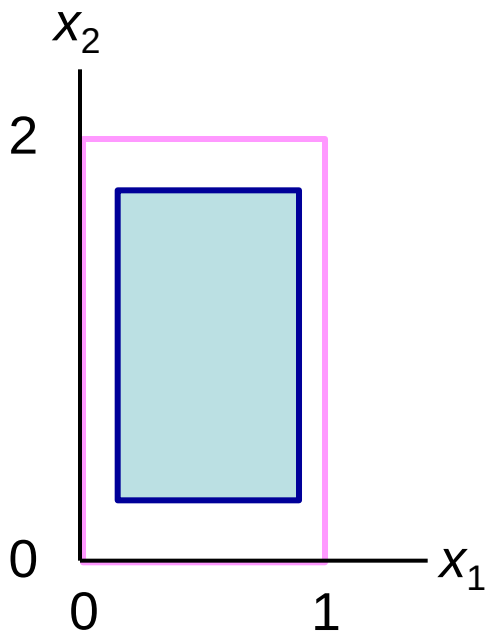
$$x_2 = \frac{1}{4x_1} \ge \frac{1}{4 \cdot 1} = 0.25$$



# Bounds propagation

- Bounds obtained by achieving bound consistency can be propagated.
  - This is important in global optimization (**range reduction**).
- Example:

$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0.125, 0.875]$$
$$x_2 \in [0.25, 1.75]$$




Propagate to constraint 2:

$$x_1 \le 1 - \frac{x_2}{2} \le \frac{0.25}{2} = 0.875$$

$$x_2 \le 2 - 2x_1 \le 2 - 2 \cdot 0.125 = 1.75$$

# Bounds propagation

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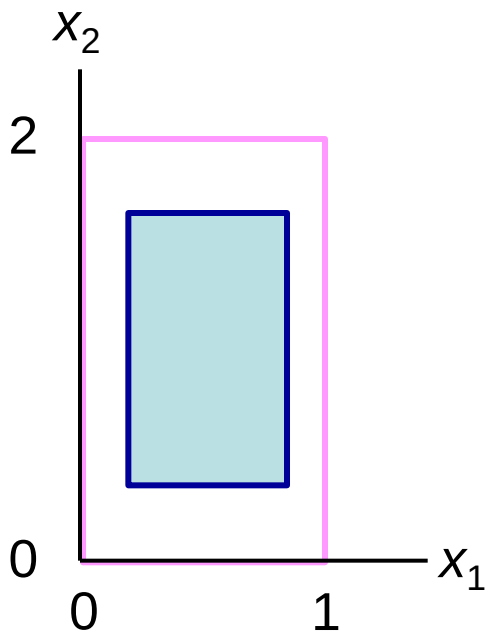
  - This is important in global optimization (**range reduction**).
- Example:

$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0.146, 0.854]$$
$$x_2 \in [0.293, 1.707]$$

Continuing, bounds asymptotically converge:



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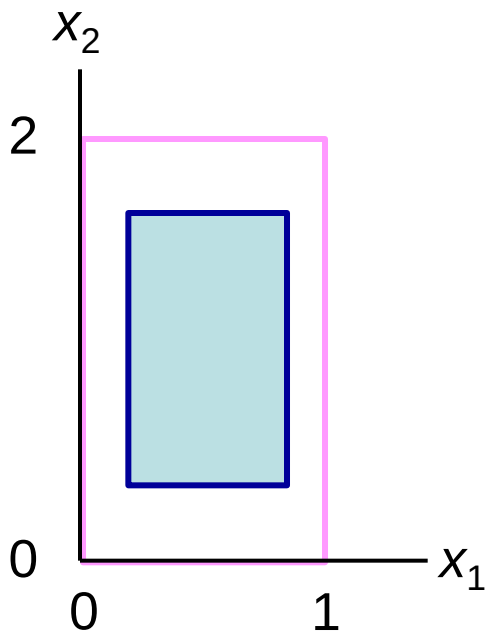
Solvers truncate the process.



# *k*-consistency

- *k*-consistency is closely related to backtracking and the **dependency graph** of a constraint set.
  - A constraint set is ***k*-consistent** if any assignment to $k - 1$ variables that violates no constraints can be extended to an assignment to $k$ variables without violating any constraints.

$$X_{j_k}$$

# *k*-consistency

- Example

$$\begin{aligned} x_1 + x_2 \quad & + x_4 \geq 1 \\ x_1 - x_2 + x_3 \quad & \geq 0 \\ x_1 \quad & - x_4 \geq 0 \\ x_j \in \{0,1\} & \end{aligned}$$

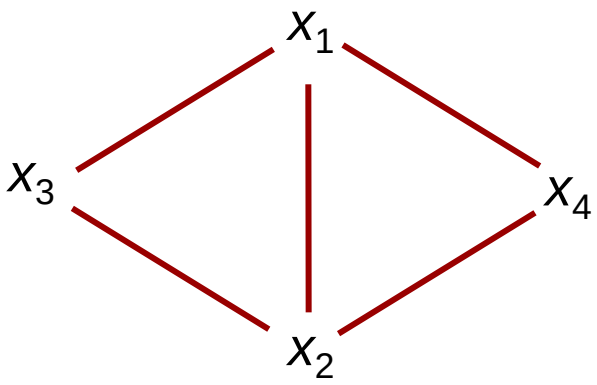
  - 2-consistent.
  - not 3-consistent:

(*x*$_1$,*x*$_2$) = (0,0) cannot be extended to (*x*$_1$,*x*$_2$,*x*$_4$) = (0,0,?).

# Dependency graph

• **Dependency graph**: variables are connected by edges when they occur in a common constraint.

$$
\begin{aligned}
x_1 + x_2 \quad &+ x_4 \geq 1 \\
x_1 - x_2 + x_3 \quad &\geq 0 \\
x_1 \quad &- x_4 \geq 0 \\
x_j \in \{0,1\}
\end{aligned}
$$




Dependency graph for the example.

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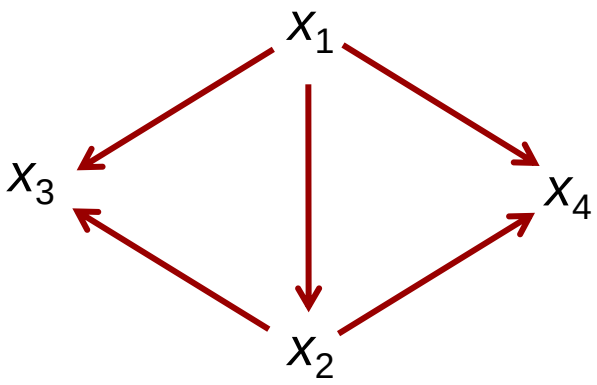



For a given variable ordering, **width** of the graph is the maximum in-degree

Here, width = 2
for ordering 1,2,3,4

# Backtracking

• A constraint set is **strongly** *k*-consistent if it is *i*-consistent for $i = 1,\ldots,k$.

**Theorem** (Freuder). If the dependency graph has width < *k* for some variable ordering, then branching (in that order) solves a strongly *k*-consistent problem **without backtracking**.

# Backtracking

- The example doesn't satisfy the conditions of the theorem.
  - Width = 2, not strongly 3-consistent.
  - Backtracking occurs when we set

  $(x_1,x_2,x_3,x_4) = (0,0,0,?)$

$$
\begin{aligned}
x_1 + x_2 \qquad + x_4 &\geq 1 \\
x_1 - x_2 + x_3 \qquad &\geq 0 \\
x_1 \qquad - x_4 &\geq 0 \\
x_j &\in \{0,1\}
\end{aligned}
$$

# Backtracking

- Suppose we add two constraints:
  - This is strongly 3-consistent.
  - Backtracking does not occur.

$$
\begin{aligned}
x_1 + x_2 \quad\quad + x_4 &\geq 1 \\
x_1 - x_2 + x_3 \quad\quad &\geq 0 \\
x_1 \quad\quad - x_4 &\geq 0 \\
x_1 + x_2 \quad\quad &\geq 1 \\
x_1 \quad + x_3 \quad\quad &\geq 1 \\
x_j &\in \{0,1\}
\end{aligned}
$$

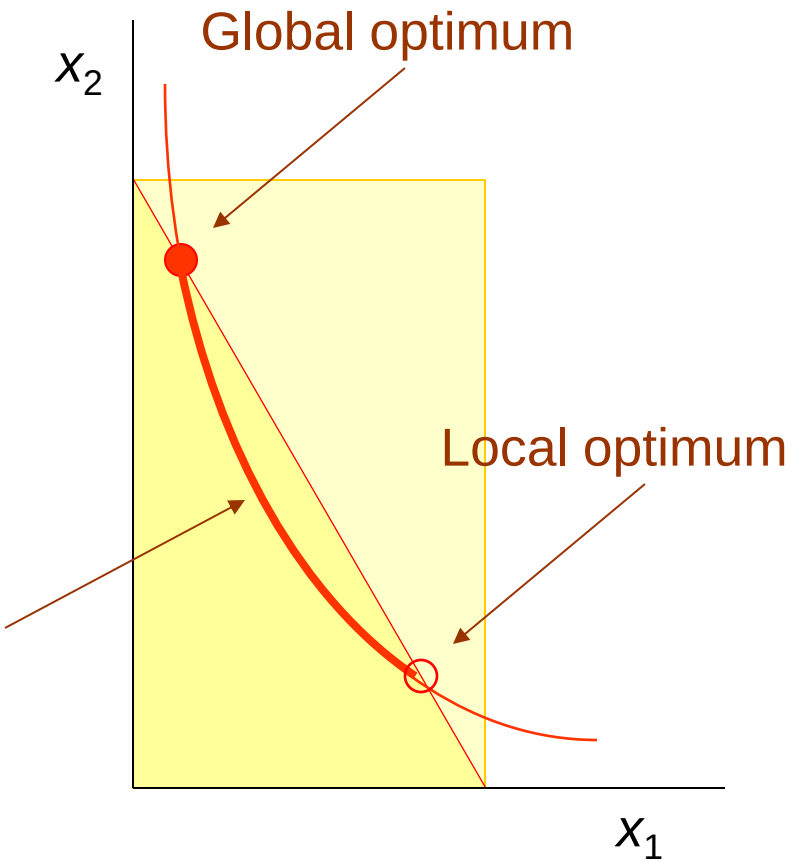
# Backtracking

- Suppose we add two constraints:
  - This is strongly 3-consistent.
  - Backtracking does not occur.
- These are valid cuts!
  - Cuts reduce backtracking by increasing the degree of consistency as well as by strengthening the LP relaxation.

$$
\begin{aligned}
x_1 + x_2 \qquad\quad + x_4 &\geq 1 \\
x_1 - x_2 + x_3 \qquad\quad &\geq 0 \\
x_1 \qquad\quad - x_4 &\geq 0 \\
x_1 + x_2 \qquad\quad &\geq 1 \\
x_1 \qquad + x_3 \qquad\quad &\geq 1 \\
x_j \in \{0,1\}
\end{aligned}
$$

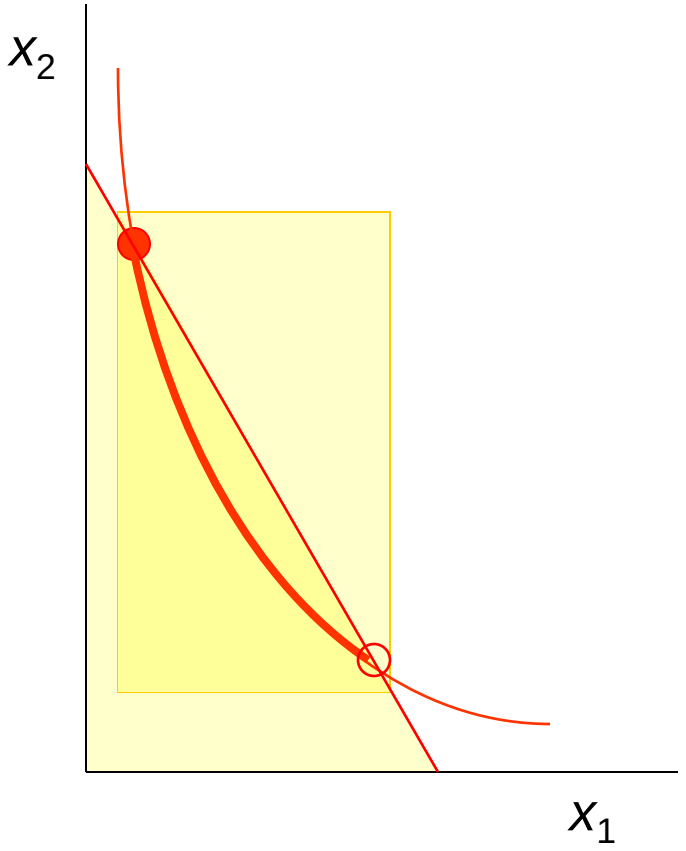
# Global optimization example

$$\max \; x_1 + x_2$$
$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0,1], \quad x_2 \in [0,2]$$



# Interval propagation (range reduction)

Propagate intervals
[0,1], [0,2]
through constraints
to obtain
[1/8,7/8], [1/4,7/4]



# Relaxation (function factorization)

Factor complex functions into elementary functions that have known linear relaxations (**McCormick factors**).

Write $4x_1x_2 = 1$ as $4y = 1$ where $y = x_1x_2$.

This factors $4x_1x_2$ into linear function $4y$ and bilinear function $x_1x_2$.

Linear function $4y$ is its own linear relaxation.

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Bilinear function $y = x_1x_2$ has relaxation:

$$\underline{x}_2x_1 + \underline{x}_1x_2 - \underline{x}_1\underline{x}_2 \le y \le \underline{x}_2x_1 + \overline{x}_1x_2 - \overline{x}_1\underline{x}_2$$
$$\overline{x}_2x_1 + \overline{x}_1x_2 - \overline{x}_1\overline{x}_2 \le y \le \overline{x}_2x_1 + \underline{x}_1x_2 - \underline{x}_1\overline{x}_2$$

where domain of $x_j$ is $[\underline{x}_j, \overline{x}_j]$

# Relaxation (function factorization)

The linear relaxation becomes:

$$\min \; x_1 + x_2$$

$$4y = 1$$

$$2x_1 + x_2 \le 2$$

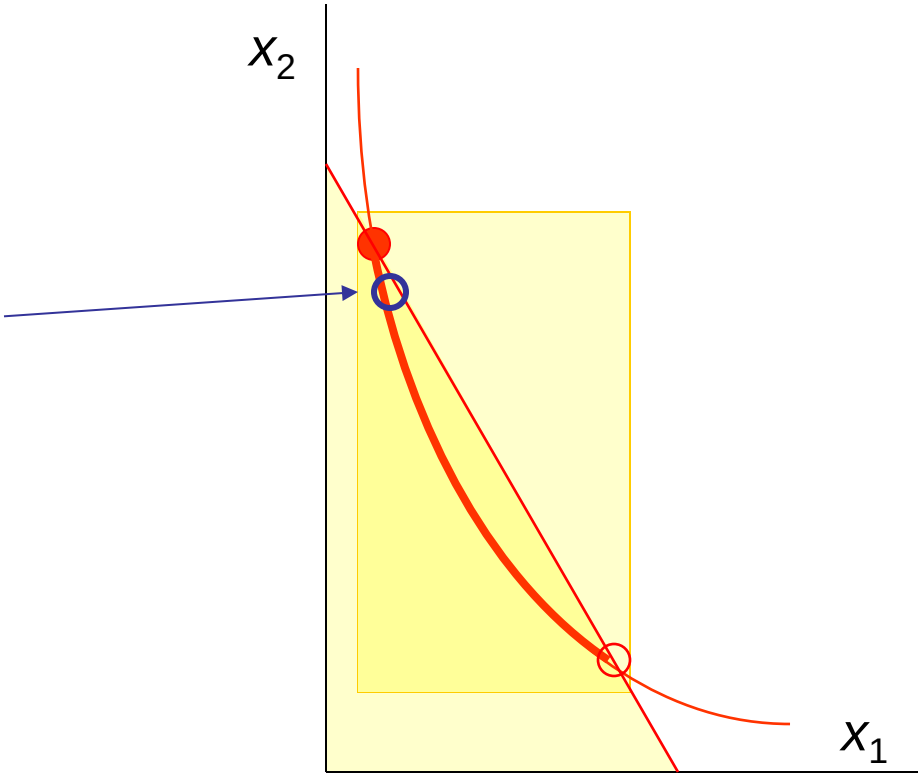
$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \overline{x}_1 x_2 - \overline{x}_1 \underline{x}_2$$

$$\overline{x}_2 x_1 + \overline{x}_1 x_2 - \overline{x}_1 \overline{x}_2 \le y \le \overline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \overline{x}_2$$

$$\underline{x}_j \le x_j \le \overline{x}_j, \;\; j = 1,2$$

# Relaxation (function factorization)

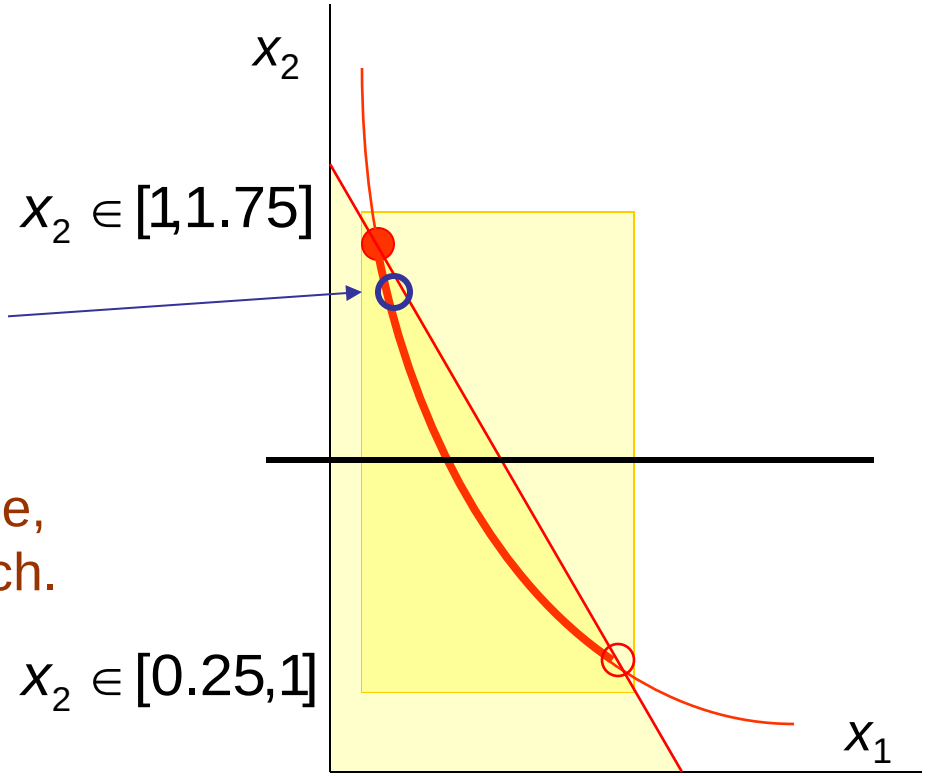
Solve linear relaxation.

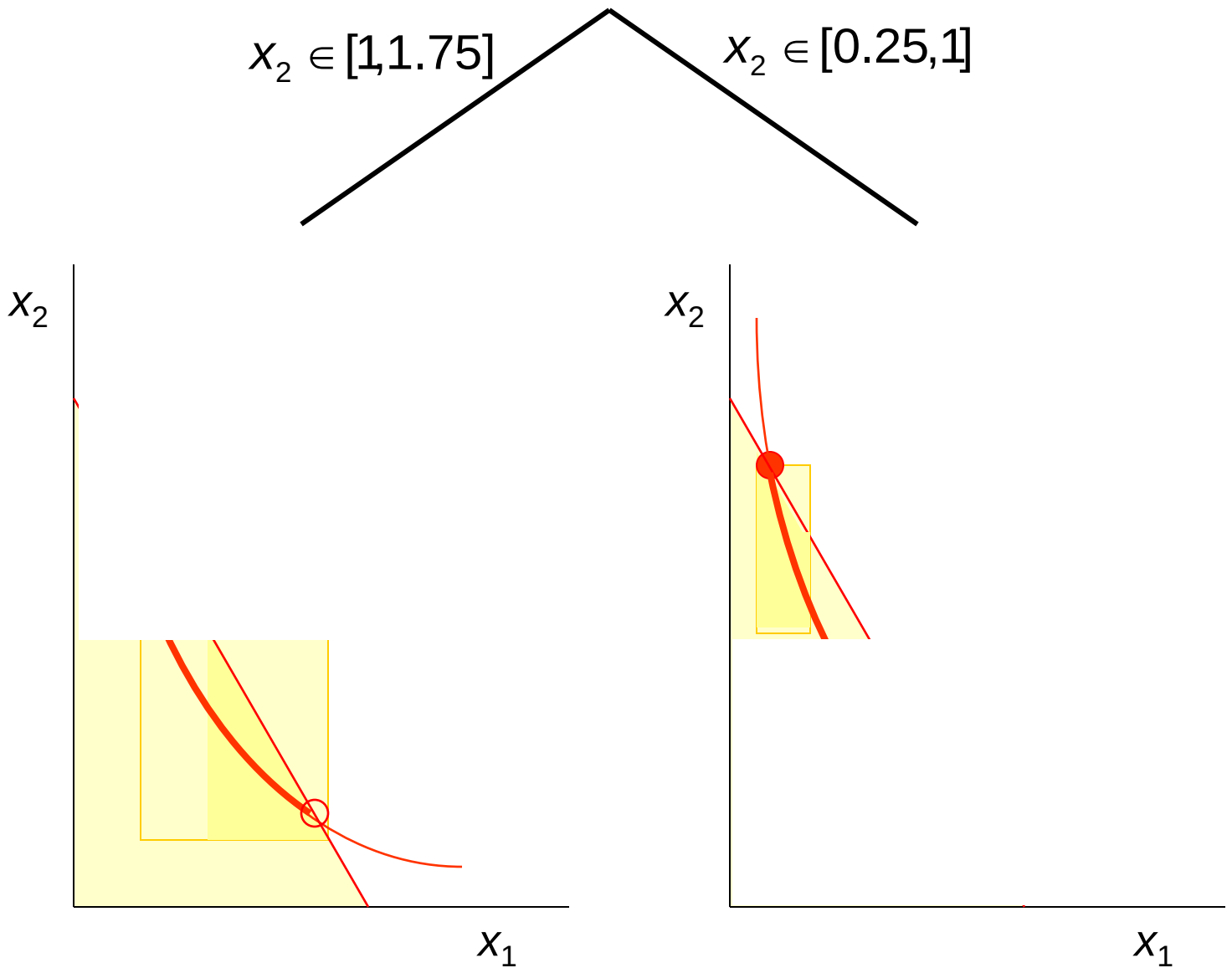


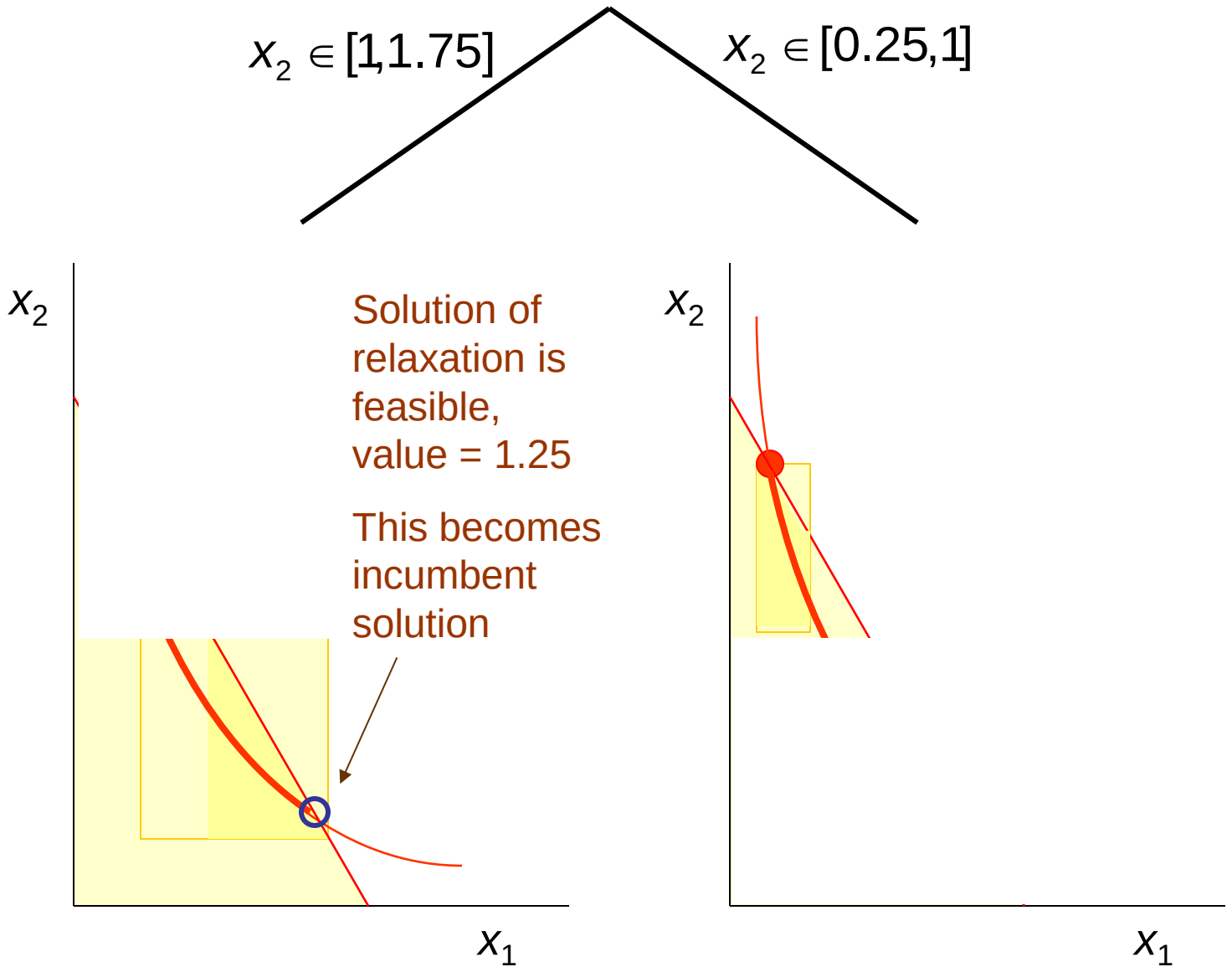
# Relaxation (function factorization)

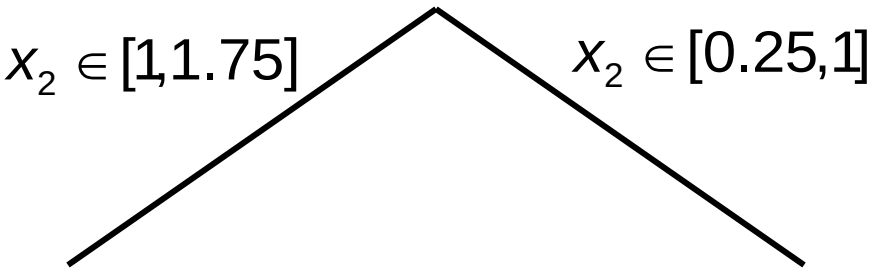
Solve linear relaxation.

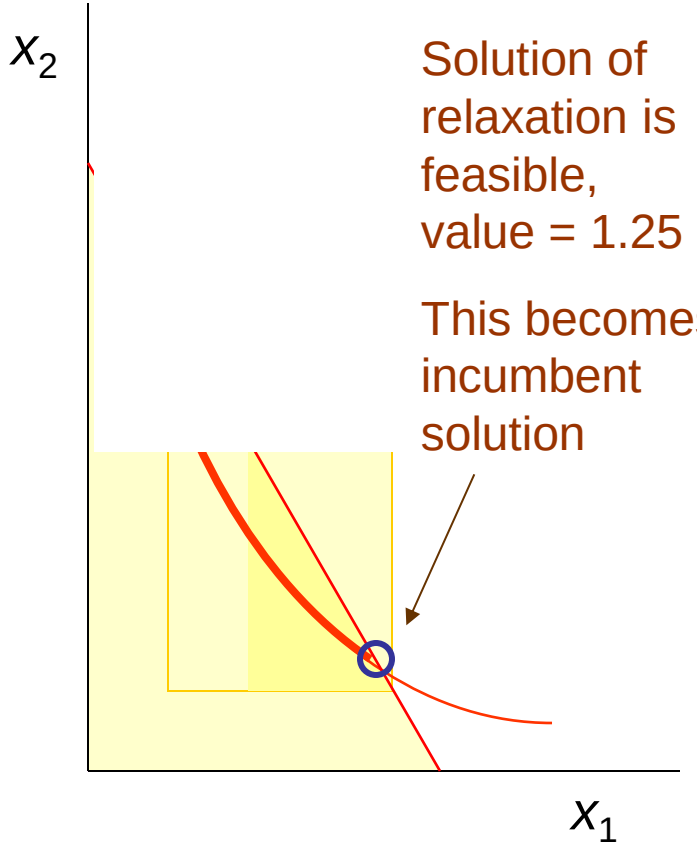
Since solution is infeasible,
split an interval and branch.




$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
$x_2$
$x_1$
$x_2$
$x_1$


$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
$x_2$
$x_1$
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
$x_2$
$x_1$




Solution of relaxation is feasible, value = 1.25

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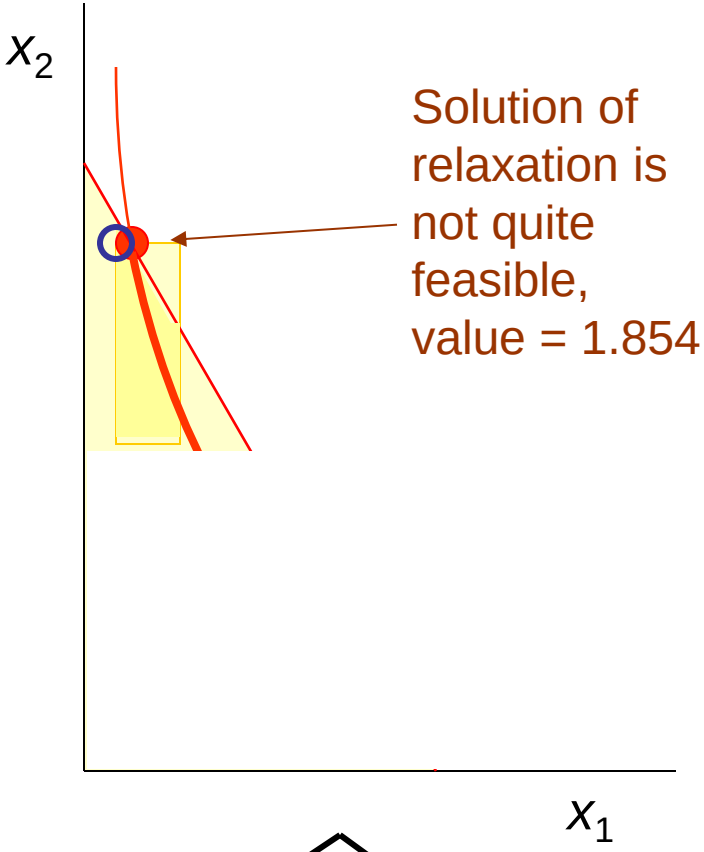
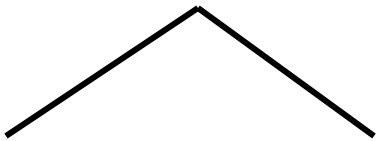



Solution of relaxation is not quite feasible, value = 1.854



# Domain filtering with Lagrange multipliers

- So far, this is all standard in global solvers.
- We can achieve stronger propagation with **filtering based on Lagrange multipliers**.
    - Reduced-cost variable fixing is a special case.

# Domain filtering with Lagrange multipliers

$$\min \; x_1 + x_2$$

$$4y = 1$$

$$2x_1 + x_2 \le 2$$

Associated Lagrange multiplier in solution of relaxation is $\lambda_2 = 1.1$

$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$

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$$\underline{x}_j \le x_j \le \bar{x}_j, \quad j = 1,2$$

This yields a valid inequality for propagation:

$$2x_1 + x_2 \ge 2 - \frac{1.854 - 1.25}{1.1} = 1.451$$

Value of relaxation

Lagrange multiplier

Value of incumbent solution

# Domain filtering with Lagrange multipliers

In general, suppose we have a relaxation:

$$\begin{aligned} &\min\ f(x) \\ &g(x) \geq 0 \\ &x \in S \end{aligned}$$

with optimal solution $x^*$, optimal value $v^*$, and **Lagrangean dual** solution $\lambda^*$.

with $\lambda_i^* > 0$, and $U$ an upper bound on the optimal value of the original problem (perhaps from an incumbent solution).

Then we have the inequality $g_i(x) \leq \dfrac{U - v^*}{\lambda_i^*}$

…which can be propagated.

# Domain filtering with Lagrange multipliers

A special case applies to **individual variables**:

$$\begin{aligned} &\min\ f(x) \\ &g(x) \geq 0 \\ &x \in S \end{aligned}$$

has optimal solution $x^*$, optimal value $v^*$, and **reduced gradient** $r$.

with $x_j^* = 0$, and $U$ an upper bound on the optimal value of the original problem (perhaps from an incumbent solution).

Then we have the inequality $x_j \leq \dfrac{U - v^*}{r_j}$

…which fixes $x_j = 0$ if bound < 1 and $x_j$ is integer
(**reduced cost variable fixing**)

# Piecewise linear functions

- Piecewise linear approximation is a powerful tool for nonlinear optimization.
  - Particularly if nonlinearities are additively separable: $f(x) = \sum_j f_j(x_j)$
- However, MINLP models require auxiliary variables.
  - A serious limitation.

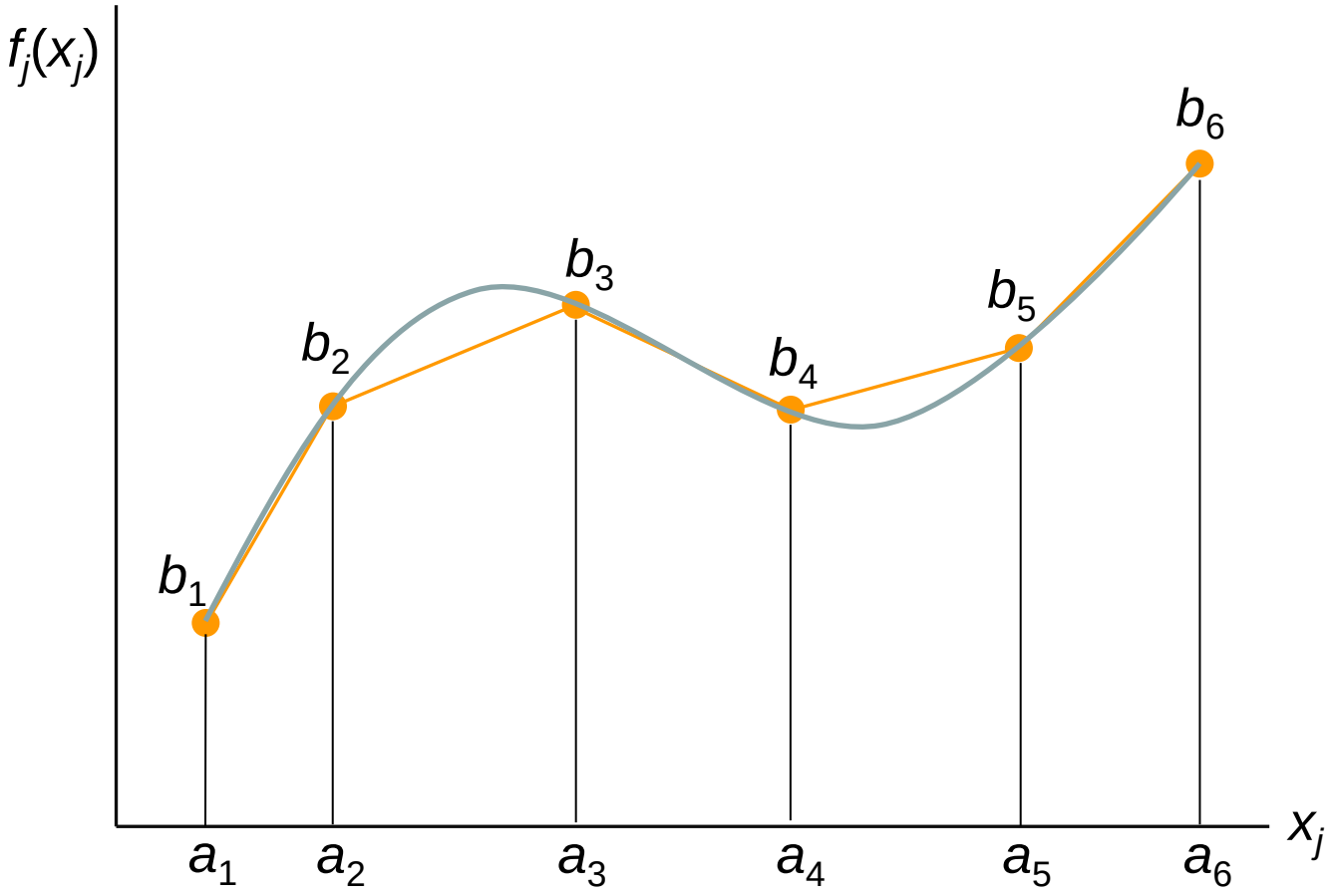
# Piecewise linear functions

- CP approach **adds no variables**.
  - while providing convex hull relaxation (tight as any locally ideal MILP model)
- Use piecewise linear **global constraint** for $f_j(x_j)$:

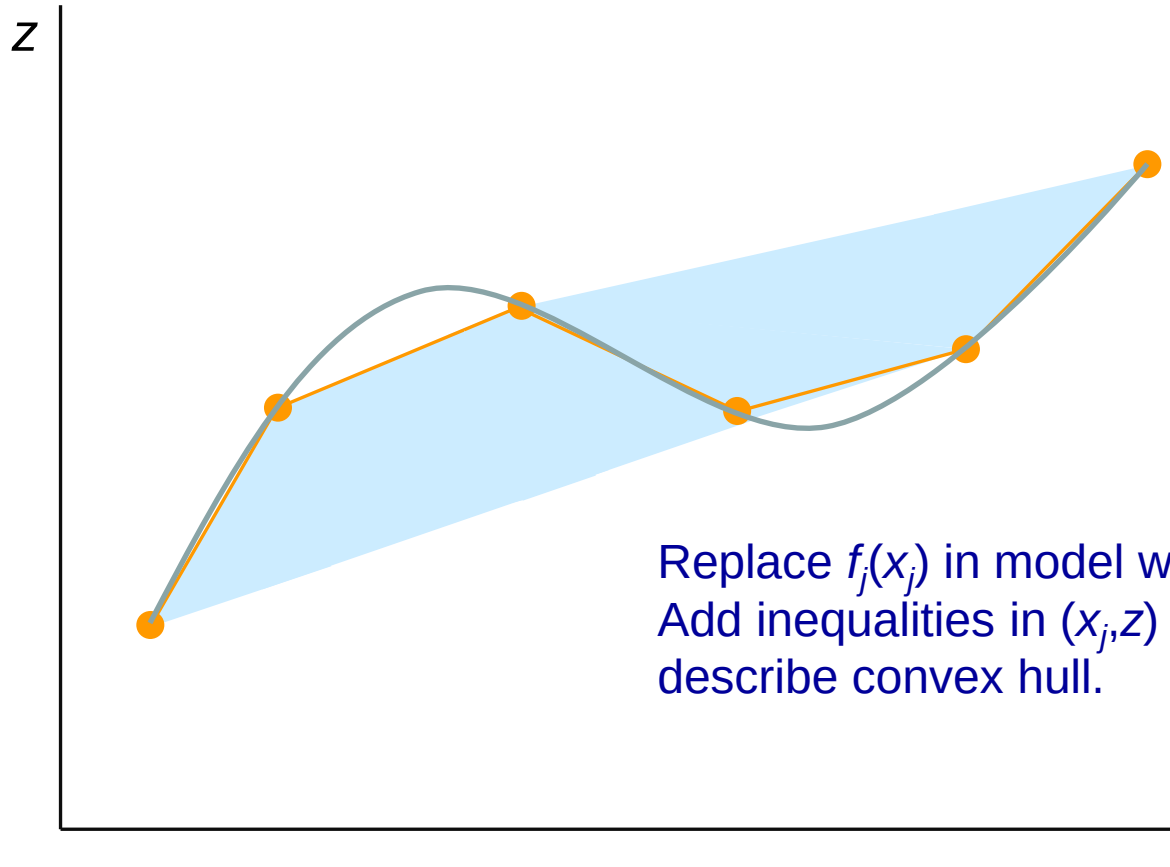
$$\text{piecewise}\left(f_j, x_j, a, b\right)$$

  - where breakpoints are $a = (a_1, \ldots a_n)$ with $f_j(a_i) = b_i$.

# Piecewise linear functions



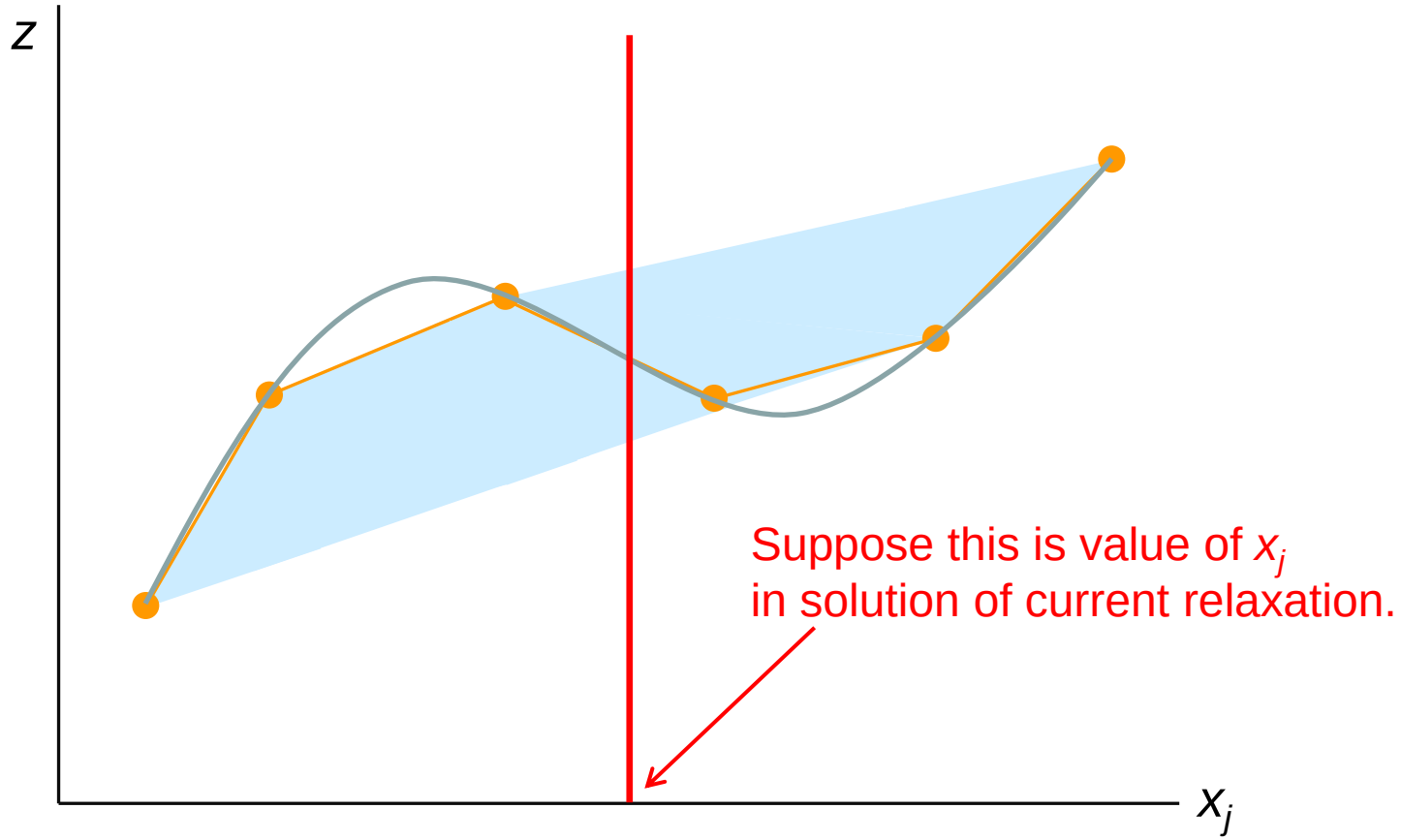
# Piecewise linear functions



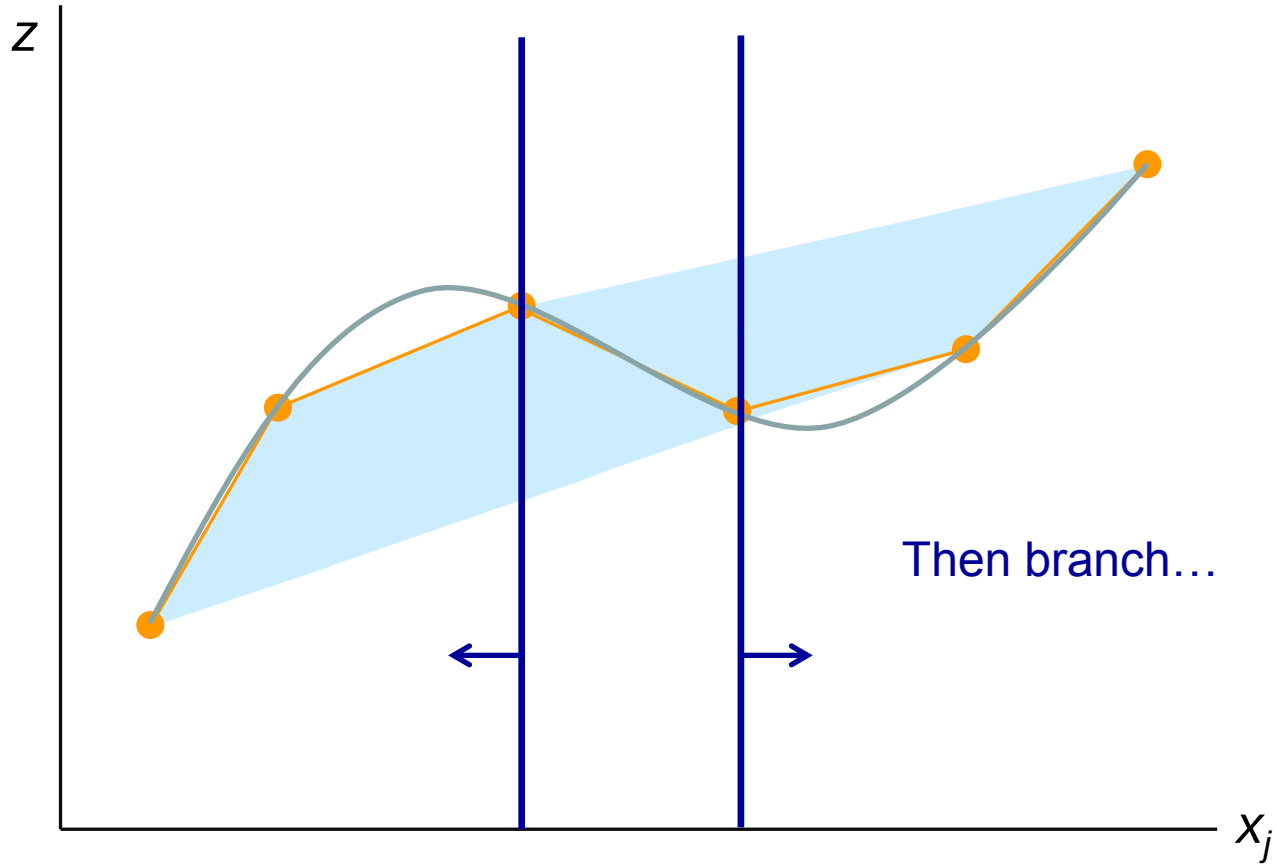

Replace $f_j(x_j)$ in model with $z$.
Add inequalities in $(x_j,z)$ that describe convex hull.

Convex hull can be computed very rapidly.

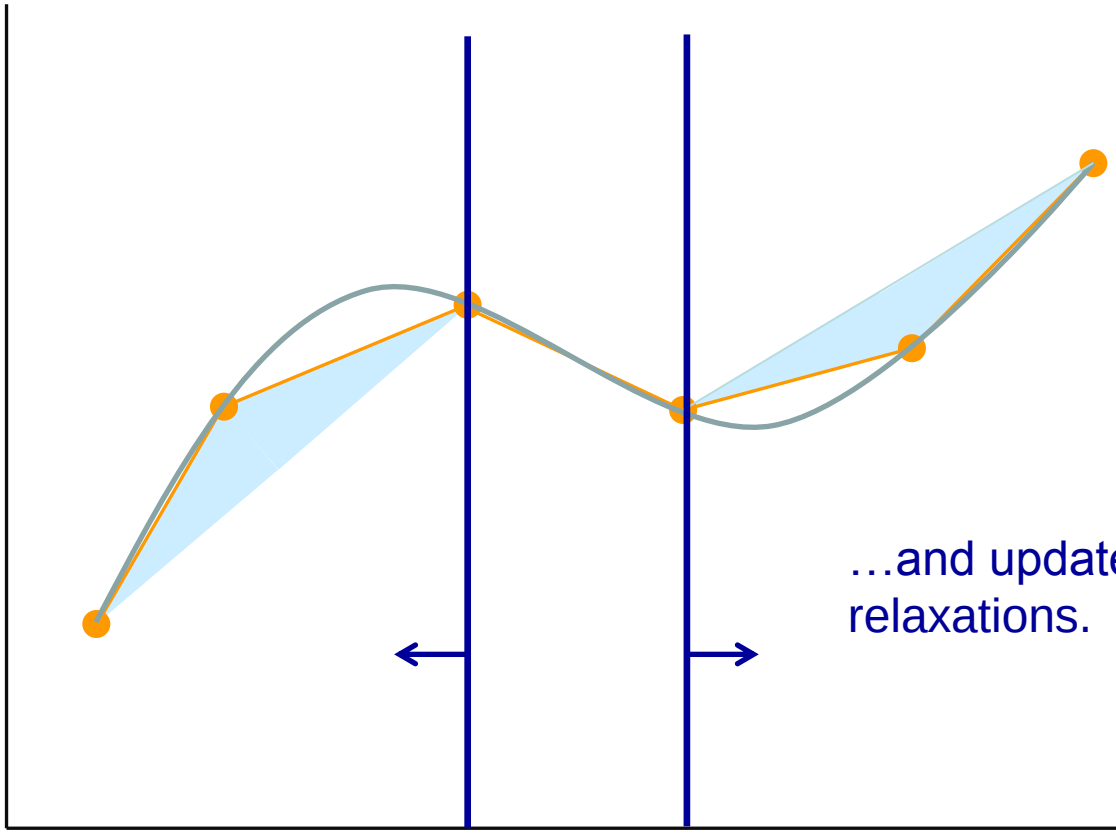
# Piecewise linear functions



# Piecewise linear functions

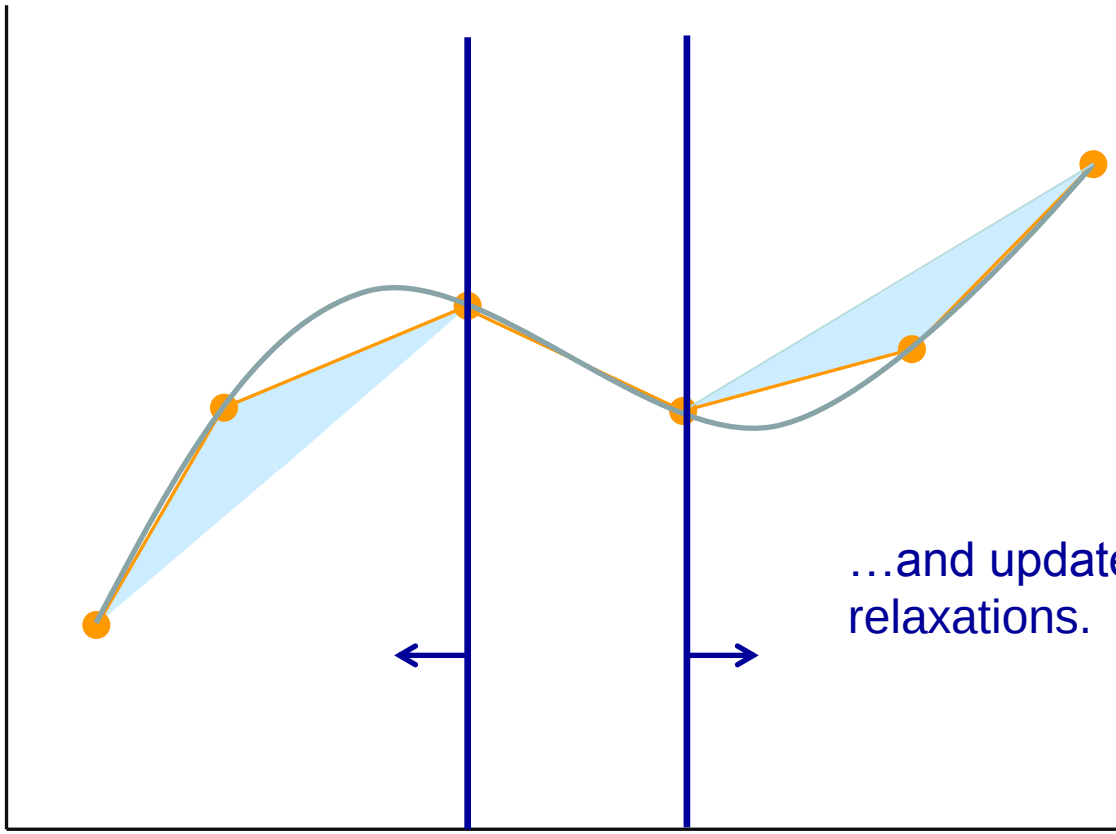


# Piecewise linear functions



…and update convex hull relaxations.

# Piecewise linear functions



…and update convex hull relaxations.

Easily extended to functions $f_j(x_i,x_k)$
By computing 3D convex hull.

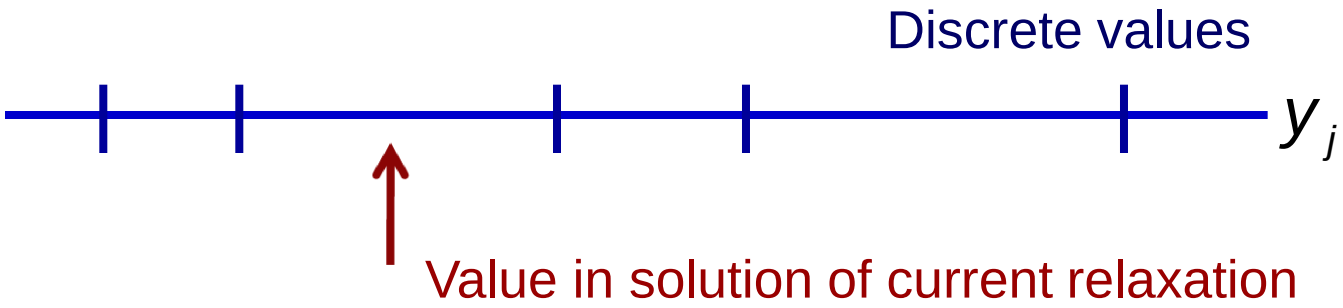
# Branching

- In general, branch on discrete values of a variable.
  - …rather than introduce 0-1 variables to model discrete values.
- For a troublesome continuous variable, **discretize it** and branch.
  - Use **many break points** without increasing size of model.

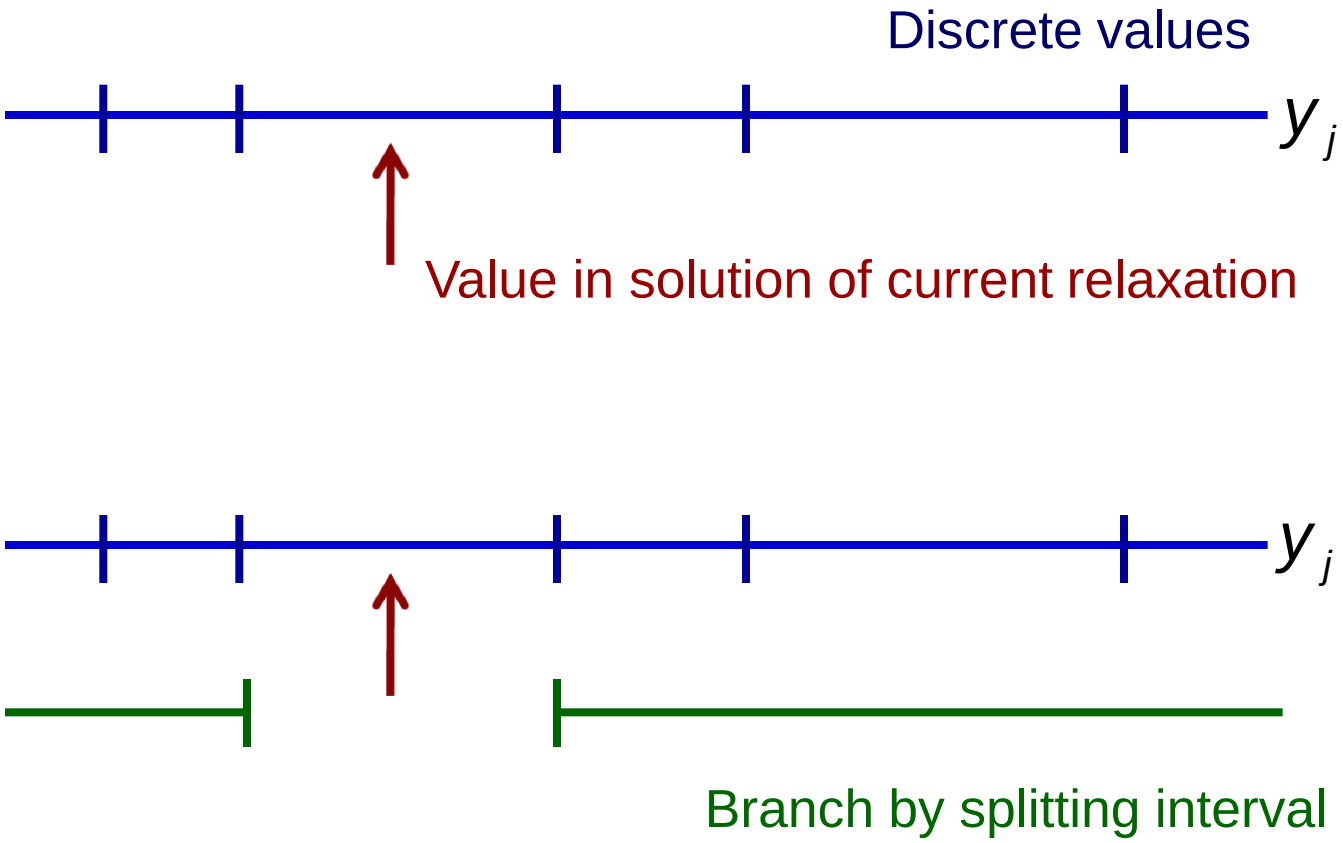
# Branching

- In general, branch on discrete values of a variable.
  - …rather than introduce 0-1 variables to model discrete values.
- For a troublesome continuous variable, **discretize it** and branch.
  - Use **many break points** without increasing size of model.
- This may allow for a **convex "**relaxation" (actually, **quasi-relaxation**)
  - If the model becomes convex when discretized variables are fixed.
  - A quasi-relaxation is not a valid relaxation but yields a valid bound on the objective function.

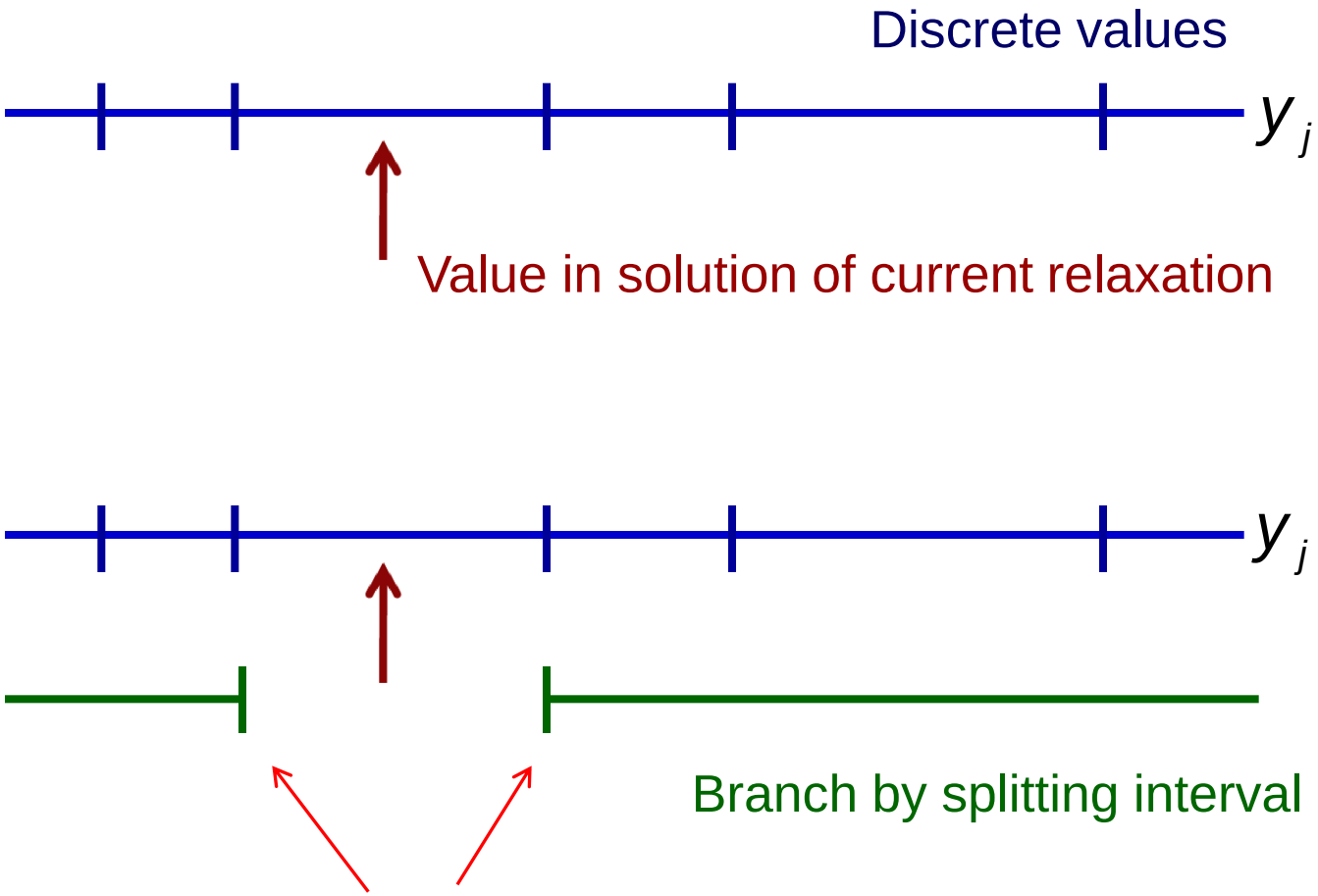
# Branching


Discrete values
$y_j$
Value in solution of current relaxation

# Branching


Discrete values
$y_j$
Value in solution of current relaxation
$y_j$
Branch by splitting interval

# Branching




Solution of next relaxation likely to be at an endpoint.
This branching intelligence unavailable in 0-1 model.

# Quasi-relaxation

Given problem $\min_{x \in S} \{ f(x) \}$

The problem $\min_{x \in S'} \{ f'(x) \}$ is a **quasi-relaxation** if

for any $x \in S$, there is an $x' \in S'$ with $f'(x') \leq f(x)$.

A quasi-relaxation need not be a valid relaxation.

But its **optimal value** is a **valid lower bound** on the optimal value of the original problem.

# Quasi-relaxation

Consider the problem

$$\min\ f(x)$$
$$g^j(x, y_j) \le 0, \text{ all } j$$
$$x \in \mathbb{R}^n,\ y_j \text{ discrete}$$

# Quasi-relaxation

Consider the problem $\min f(x)$

$$g^j(x, y_j) \le 0, \text{ all } j$$

$$x \in \mathbb{R}^n, y_j \text{ discrete}$$

Each $g^j$ is a vector of functions

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Each $g^j$ is a vector of functions

Each $y_j$ is a scalar variable

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Consider the problem $\min f(x)$

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$$x \in \mathbb{R}^n, y_j \text{ discrete}$$

Each $g^j$ is a vector of functions

Each $y_j$ is a scalar variable

Relaxing the problem by making $y_j$ continuous could result in a **nonconvex** problem.

# Quasi-relaxation

Consider the problem $\min f(x)$

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Relaxing the problem by making $y_j$ continuous could result in a **nonconvex** problem.

But suppose the problem becomes convex when each $y_j$ is fixed to a **constant**.

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Each $y_j$ is a scalar variable

Relaxing the problem by making $y_j$ continuous could result in a **nonconvex** problem.

But suppose the problem becomes convex when each $y_j$ is fixed to a **constant**.

Then we may be able to write a **convex quasi-relaxation**.

# Quasi-relaxation

Consider the problem

$$\min\ f(x)$$
$$g^j(x, y_j) \le 0,\ \text{all } j$$
$$x \in \mathbb{R}^n,\ y_j \text{ discrete}$$

**Theorem** (JNH)

If $f(x)$ is convex and each $g^j(x,y)$ is **semihomogeneous** and **convex** in $x$, and **concave** in scalar $y_j$, then we have a **convex quasi-relaxation**:

$$\min\ f(x)$$
$$g(x^1, y_L) + g(x^2, y_U) \le 0$$
$$\alpha x^L \le x^1 \le \alpha x^U$$
$$(1-\alpha)x^L \le x^2 \le (1-\alpha)x^U$$
$$x = x^1 + x^2,\ \alpha \in [0,1]$$

# Quasi-relaxation

Consider the problem $\min f(x)$

$$g^j(x, y_j) \le 0, \text{ all } j$$

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$$x = x^1 + x^2, \ \alpha \in [0,1]$$

$g(\alpha x, y) \le \alpha g(x,y)$ for all x, y and $\alpha \in [0,1]$,

$g(0,y) = 0$ for all $y$

# Quasi-relaxation

Consider the problem

$$\min\ f(x)$$
$$g^j(x, y_j) \le 0,\ \text{all } j$$
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Bounds on $y$

# Quasi-relaxation

Consider the problem

$$\min\ f(x)$$
$$g^j(x, y_j) \le 0, \ \text{all } j$$
$$x \in \mathbb{R}^n,\ y_j \ \text{discrete}$$

**Theorem** (JNH)

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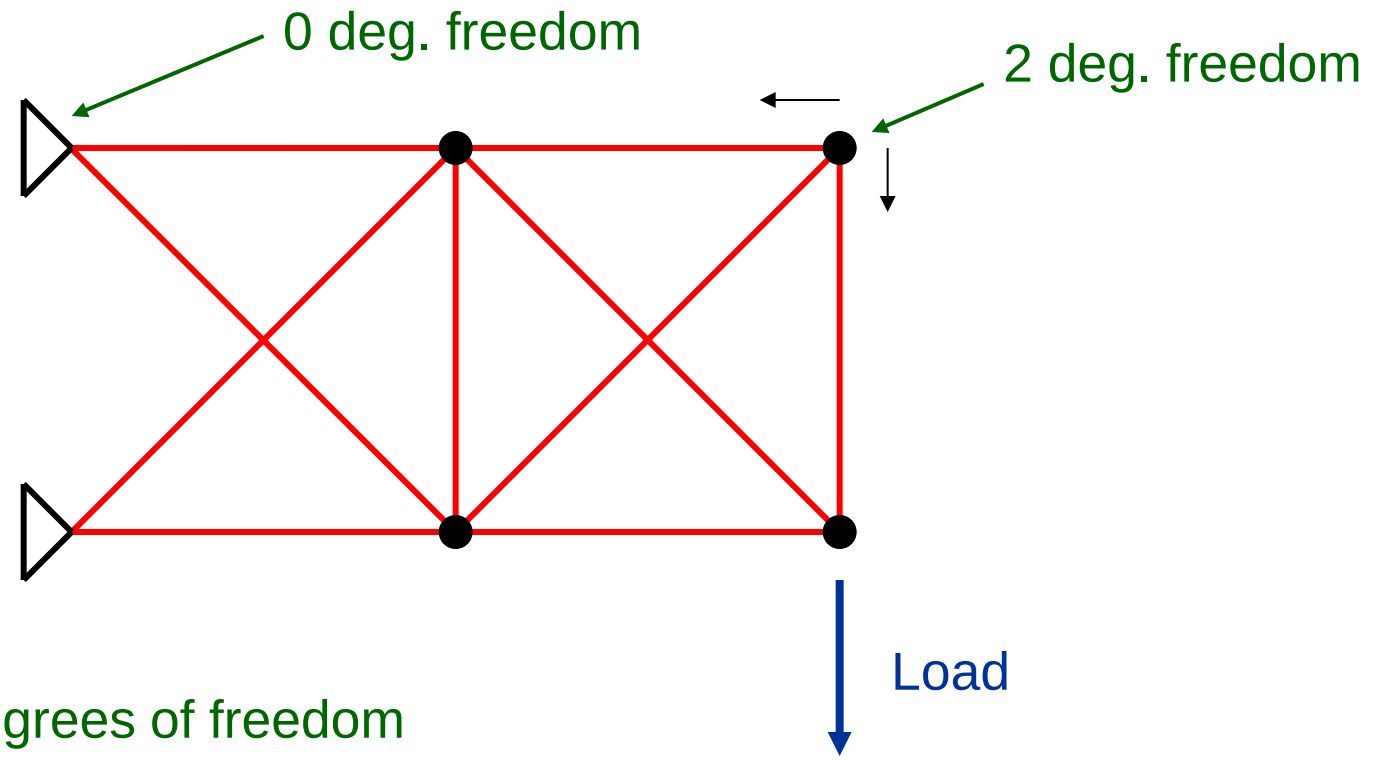
$$x = x^1 + x^2,\ \alpha \in [0,1]$$

Bounds on $x$

# Example: Truss Structure Design

Select size of each bar (possibly zero) to support the load while minimizing weight. Bar sizes are **discrete**.

10-bar cantilever truss




Total 8 degrees of freedom

# Truss Structure Design

$$\min \quad \sum_i h_i A_i$$

$$s.t. \quad \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j$$

$$\sum_j \cos\theta_{ij} d_j = v_i, \text{ all } i$$

Nonlinear, nonconvex

$$\frac{E_i}{h_i} A_i v_i = s_i, \text{ all } i$$

Hooke's law

$$v_i^L \le v_i \le v_i^U, \text{ all } i$$

$$d_j^L \le d_j \le d_j^U, \text{ all } j$$

$$\bigvee_k (A_i = A_{ik})$$

Area must be one of several discrete values $A_{ik}$

# Truss Structure Design

Can convert to MILP model by introducing new variables.

$$\min \quad \sum_i h_i \sum_k A_{ik} y_{ik}$$

$$s.t. \quad \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j$$

$$\sum_j \cos\theta_{ij} d_j = \sum_k v_{ik}, \text{ all } i$$

$$\frac{E_i}{h_i} \sum_k A_{ik} v_{ik} = s_i, \text{ all } i$$

$$v_i^L \le v_i \le v_i^U, \text{ all } i$$

$$d_j^L \le d_j \le d_j^U, \text{ all } j$$

$$\sum_k y_{ik} = 1, \text{ all } i$$

0-1 variables indicating size of bar *i*

Elongation variable disaggregated by bar size

Hooke's law becomes linear

# Quasi-relaxation

$$\min\ f(x)$$
$$g^j(x, y_j) \le 0,\ \text{all } j$$
$$x \in \mathbb{R}^n,\ y_j \text{ discrete}$$

$\frac{E_i}{h_i} A_i v_i = s_i$ has the form $g(x,y_j) = 0$ with $g$ semihomogeneous in $x$ and concave (linear) in $y_j$ because we can write it

$$\frac{E_i}{h_i} A_i v_i - s_i = 0$$

with $x = (A_i, s_i)$, $y_j = v_i$.

# Truss Structure Design

So we have a quasi-relaxation of the truss problem:

$$\min \quad \sum_i h_i [A_i^L y_i + A_i^U (1 - y_i)]$$

$$s.t. \quad \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j$$

$$\sum_j \cos\theta_{ij} d_j = v_{i0} + v_{i1}, \text{ all } i$$

$$\frac{E_i}{h_i}(A_i^L v_{i0} + A_i^U v_{i1}) = s_i, \text{ all } i$$

$$v_i^L y_i \le v_{i0} \le v_i^U y_i, \text{ all } i$$

$$v_i^L (1 - y_i) \le v_{i1} \le v_i^U (1 - y_i), \text{ all } i$$

$$d_j^L \le d_j \le d_j^U, \text{ all } j$$

$$0 \le y_i \le 1, \text{ all } i$$

Hooke's law is linearized

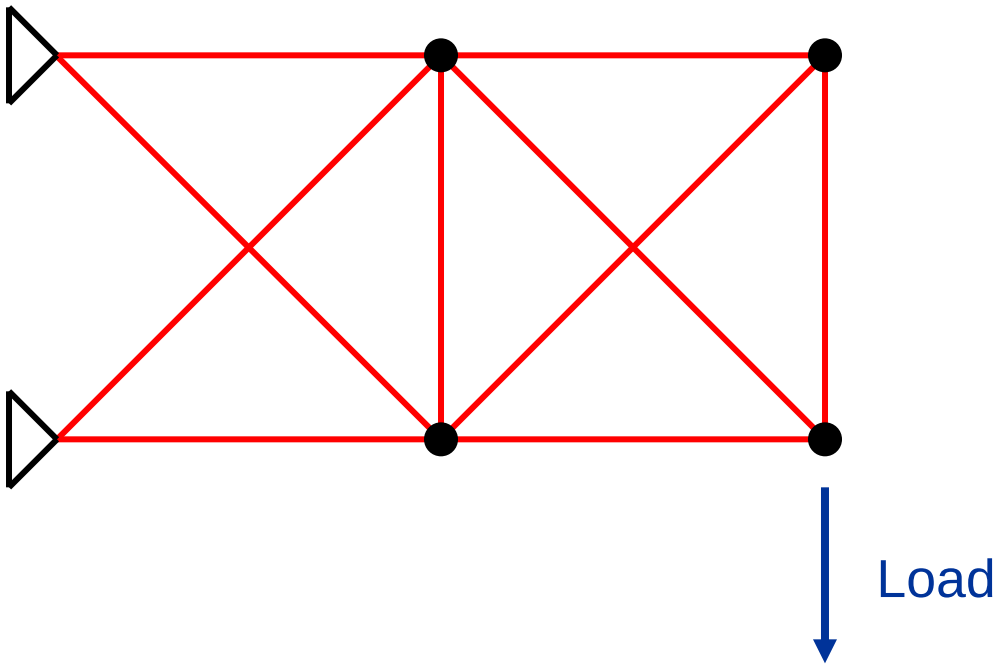
Elongation bounds split into 2 sets of bounds

# Truss Structure Design

Some computational results…

10-bar cantilever truss

Yunes, Aron, JNH (2010),
*based on*
Bollapragada, Ghattas, JNH (2001)



# Truss Structure Design

SIMPL = integrated solver that implements CP-style branching and quasi-relaxations

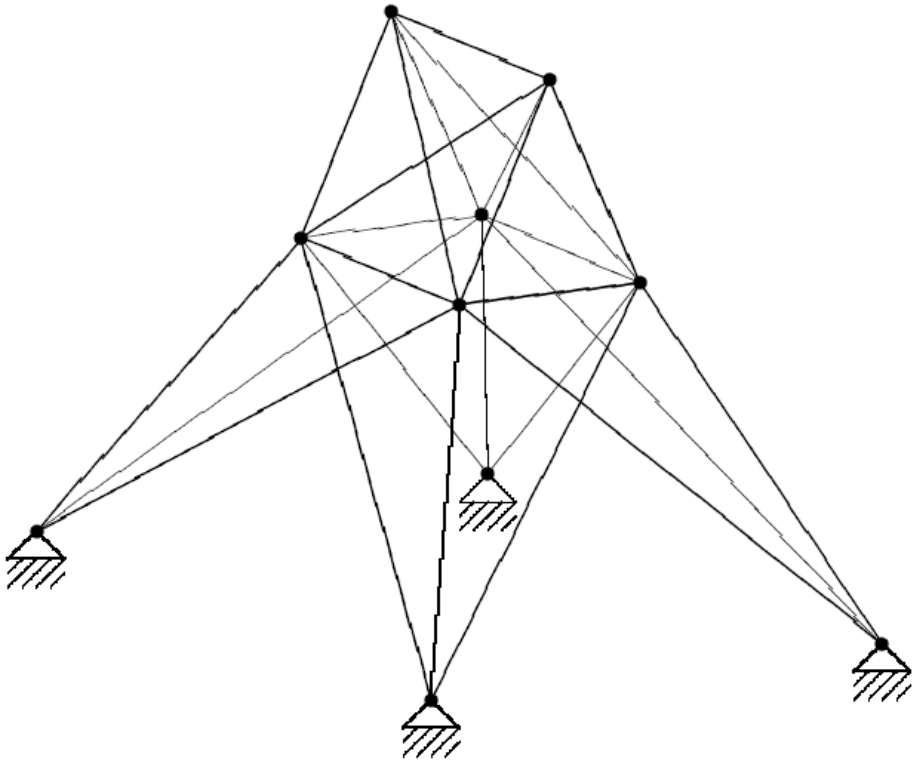
## Computational results (seconds)

| No. bars | Loads | BARON | CPLEX | SIMPL |
|---|---|---|---|---|
| 10 | 1 | 5.3 | 0.40 | 0.08 |
| 10 | 1 | 3.8 | 0.26 | 0.07 |
| 10 | 1 | 8.1 | 0.83 | 0.49 |
| 10 | 1 | 8.8 | 1.2 | 0.63 |
| 10 | 2 | 24 | 4.9 | 1.84 |
| 10 | 2* | 327 | 146 | 65 |
| 10 | 2* | 2067 | 1087 | 651 |

*plus displacement bounds

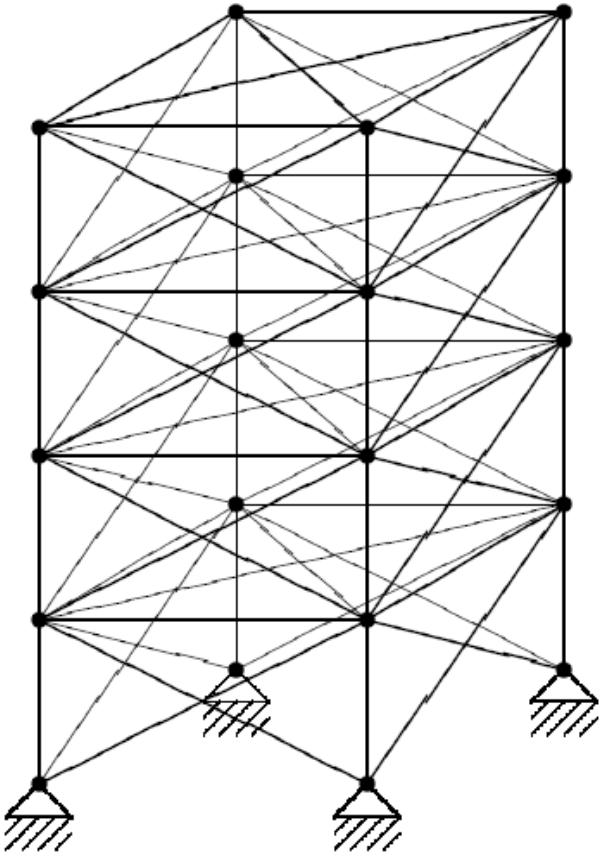
# Truss Structure Design

25-bar problem



# Truss Structure Design

72-bar problem



# Truss Structure Design

## Computational results (seconds)

| No. bars | Loads | BARON | CPLEX | SIMPL |
|---|---|---|---|---|
| 25 | 2 | 3,302 | 44 | 20 |
| 72 | 2 | 3,376 | 208 | 28 |
| 90 | 2 | 21,011 | 570 | 92 |
| 108 | 2 | > 24 hr* | 3208 | 1720 |
| 200 | 2 | > 24 hr* | > 24 hr* | > 24 hr** |

* no feasible solution found

** best feasible solution has cost 32,700

# Decision diagrams

- A **decision diagram** can represent the feasible set of a discrete optimization problem.
  - An optimal solution is a **shortest path** in the diagram.
  - Linearity, convexity **irrelevant**.
  - Provide **enhanced propagation** in a CP context.
- Proposal: **discretize** continuous variables and optimize over a decision diagram.
  - Branching in **relaxed** decision diagrams may permit **massive discretization**.
  - A “big data” technique.

# Decision diagrams

- The knapsack constraint

$$300x_0 + 300x_1 + 285x_2 + 285x_3 + 265x_4 + 265x_5 + 230x_6 + 23x_7 + 190x_8 + 200x_9 + 400x_{10} + 200x_{11} + 400x_{12} + 200x_{13} + 400x_{14} + 200x_{15} + 400x_{16} + 200x_{17} + 400x_{18} \geq 2701$$

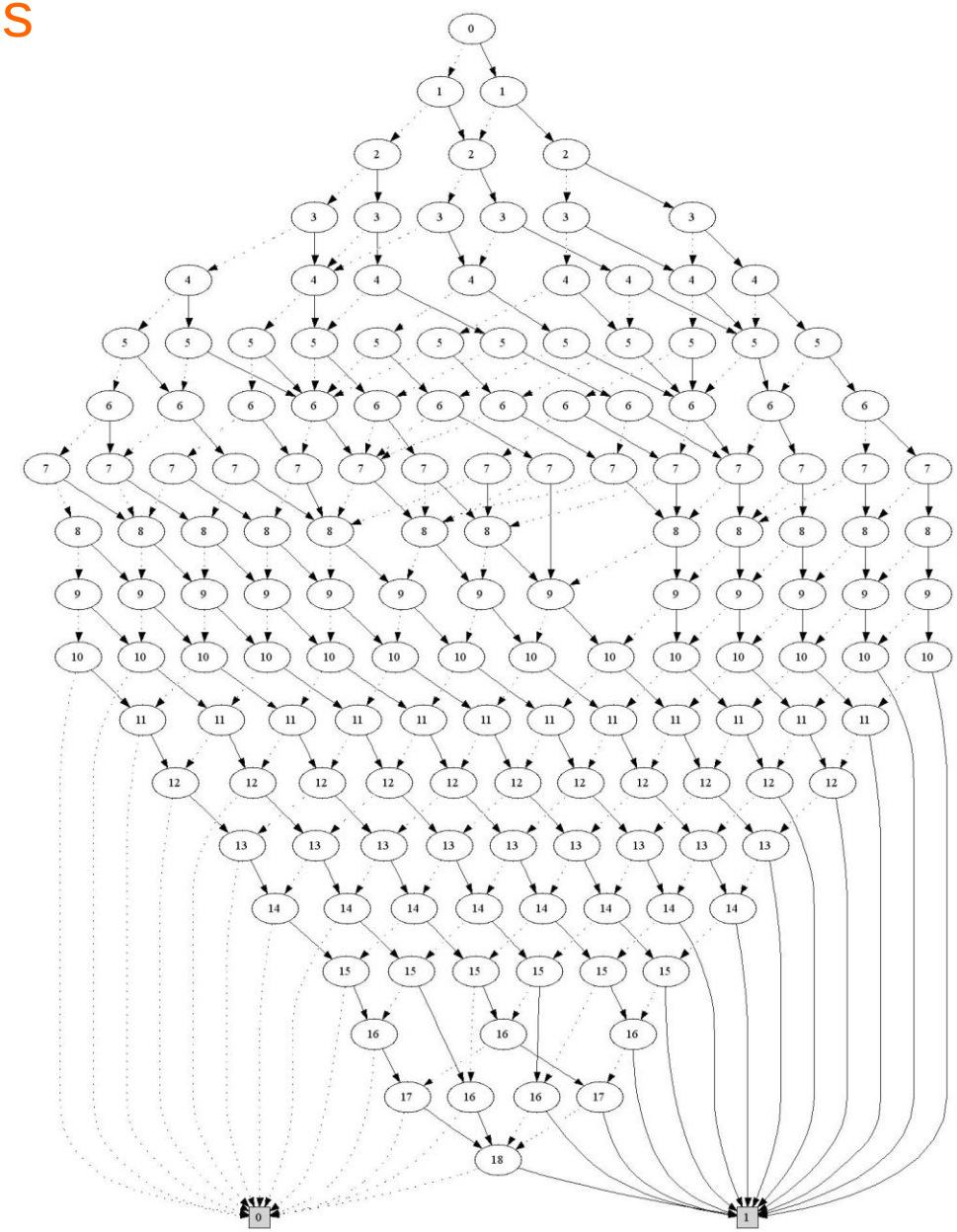
has 117,520 minimal feasible solutions.

- But its reduced decision diagram has only 152 nodes…

# Decision diagrams

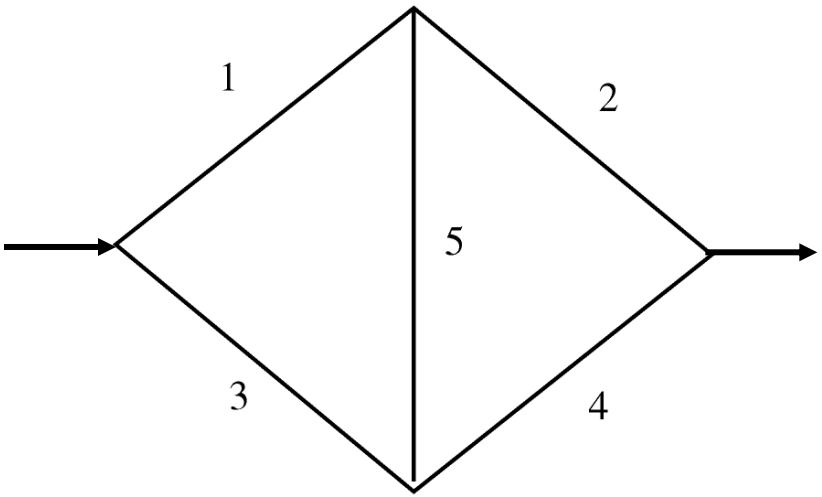
A branch from layer *i* represents fixing $x_i$ to 0 (dashed arc) or 1 (solid arc).

Paths to 1 correspond to feasible solutions.



# Example: network reliability

- Minimize cost subject to a bound on reliability (highly nonconvex)
  - System of 5 bridges:




$$R = R_1R_2+(1-R_2)R_3R_4+(1-R_1)R_2R_3R_4+R_1(1-R_2)(1-R_3)R_4R_5+(1-R_1)R_2R_3(1-R_4)R_5$$

# Example: network reliability

The problem:

$$\min \sum_j c_j x_j$$

Number of links at bridge $j$

$$R \geq R_{\min}$$

$$R = R_1 R_2 + (1-R_2) R_3 R_4 + (1-R_1) R_2 R_3 R_4$$
$$+ R_1 (1-R_2)(1-R_3) R_4 R_5 + (1-R_1) R_2 R_3 (1-R_4) R_5$$

$$R_j = 1 - (1 - r_j)^{x_j}, \text{ all } j$$

$$x_j \in \{0,1,2,3\}$$

Reliability of one link for bridge $j$

Set min desired reliability to $R_{\min}$ = 60%

Eliminate variables $R_i$, leaving one **continuous** variable $R$.
**Discretize** $R$ for the decision diagram.

# Example: network reliability

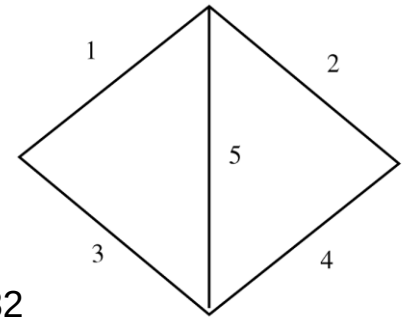
Optimal solution

| $c_{opt} + \Delta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $R$ |
|---|---|---|---|---|---|---|
| 50: | 0 | 0 | 1 | 1 | 0 | 72 |
| 60: | 1 | 1 | 0 | 0,2 | | 79 |
| 85: | 2 | | | | | 84 |
| 90: | | | 2 | 3 | | 86 |
| 95: | | 2 | | | 1 | 88 |
| 100: | | | | | | 95 |
| 120: | | | | | | 97 |
| 125: | 3 | | | | | |
| 155: | | 3 | | | 2 | |
| 160: | | | | | | 98 |
| 170: | | | | | | 99 |
| 180: | | | 3 | | | |
| 230: | | | | | 3 | |

Decision diagram has 308 nodes, generated in 1.1 sec.

Computing optimal solution is trivial (shortest path).

Bonus: we get complete **postoptimality analysis** from decision diagram



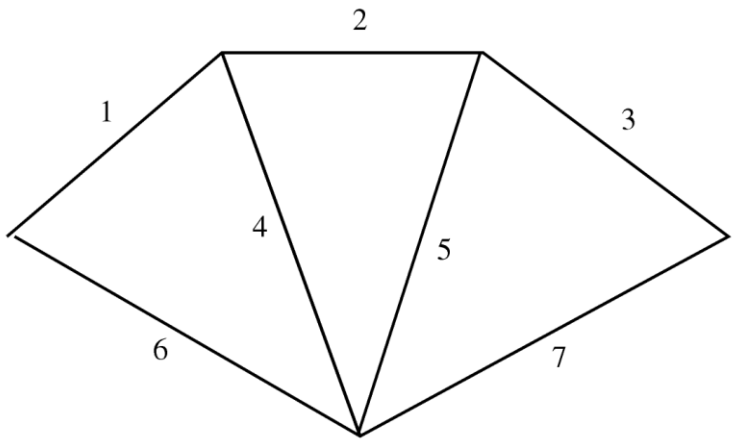

Hadzic and JNH (2006).

# Example: network reliability

Nonlinear constraints are increasingly complex for larger networks.

Decision diagram has 1779 nodes, generated in 14.8 sec.

7 bridges

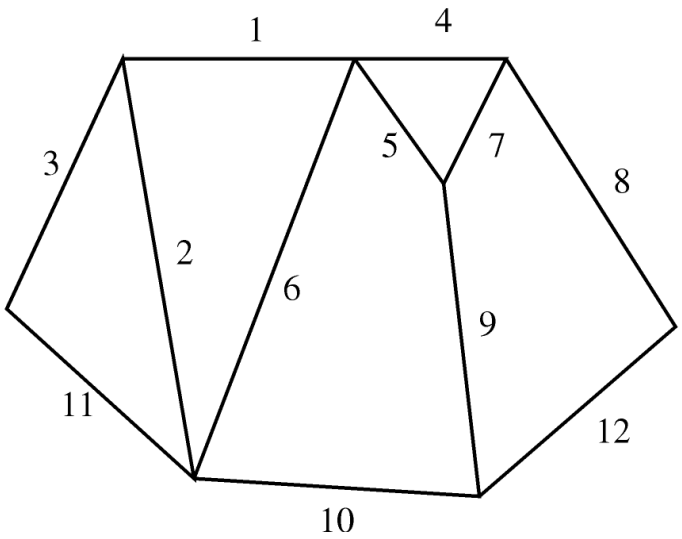



| $c_{opt}+\Delta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $R$ |
|---|---|---|---|---|---|---|---|---|
| 9: | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 72.2 |
| 11: | | | 1 | | 1 | | 0 | |
| 12: | 1 | | | 1 | | 0 | | |
| 13: | | 1 | | | | 2 | | 82.9 |
| 14: | | | | | 2 | | 2 | |
| 15: | | | 2 | 2 | | | | |
| 16: | 2 | | | | | | | |
| 17: | | | | | 3 | 3 | | 84.6 |
| 18: | | 2 | | 3 | | | | 95.2 |
| 19: | | | 3 | | | | 3 | |
| 20: | 3 | | | | | | | |
| 22: | | | | | | | | 97.2 |
| 23: | | 3 | | | | | | |
| 27: | | | | | | | | 99.2 |
| 34: | | | | | | | | 99.4 |
| 40: | | | | | | | | 99.6 |
| 43: | | | | | | | | 99.7 |
| 47: | | | | | | | | 99.8 |
| 54: | | | | | | | | 99.9 |

# Example: network reliability

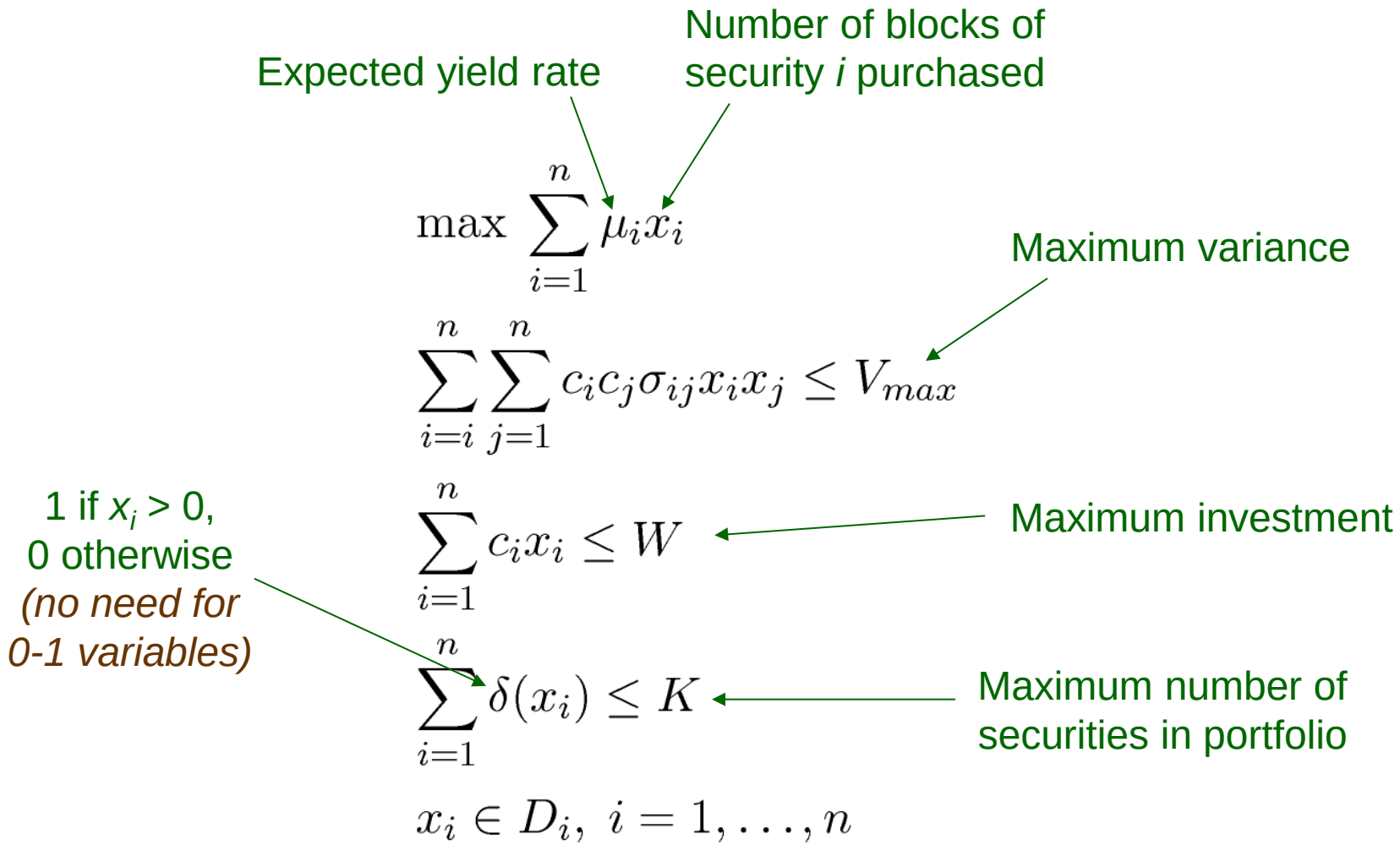
Decision diagram has 69,457 nodes, generated in 2933 sec.

12 bridges




| $c_{opt}+\Delta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $x_{11}$ | $x_{12}$ | $R$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 180 | 1 | 0 | 2 | 3 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 80 |
| 185 | | | 3 | 2 | | | | | | | | | 82 |
| 190 | | | | | | | | 3 | | | | | |
| 195 | | | | | | | | | | | | | 83 |
| 200 | | | | | | | | | | 1 | | | |
| 205 | | | | | | | | | | | | | 86 |
| 210 | 2 | | | | | | 1 | | | | | | |
| 215 | | | | | | | | | | | | | 88 |
| 220 | | | | | | | | | | 2 | | | |
| 225 | | | | | 1 | | | | 1 | | | | |
| 230 | 0 | | 0 | | | 1,2 | | | | | 1 | | |
| 235 | | | 1 | | | | | | | | | | |
| 240 | | 1 | | 1 | | | 2 | | | 3 | 2 | 1 | |
| 250 | | | | 0 | | | | 0,1 | | | | 2 | |
| 255 | | | | | | | | | | | | | 91 |
| 260 | 3 | | | | | | | | 2 | | | | |
| 265 | | | | | | | | | | | | | 93 |
| 270 | | | | | 2 | 3 | 3 | | | | | | |
| 290 | | | | | | | | | | | 3 | | |
| 300 | | 2 | | | | | | | | | | | |
| 305 | | | | | | | | | 3 | | | | |
| 310 | | | | | | | | | | | | 3 | |
| 315 | | | | | 3 | | | | | | | | 94 |
| 340 | | | | | | | | | | | | | 95 |
| 360 | | 3 | | | | | | | | | | | |
| 365 | | | | | | | | | | | | | 96 |
| 380 | | | | | | | | | | | | | 97 |
| 430 | | | | | | | | | | | | | 98 |
| 485 | | | | | | | | | | | | | 99 |

# Example: portfolio design



# Example: portfolio design

10 securities,
max 7 selected.

Decision diagram has 59,802 nodes, generated in 63 sec.

Trivial to compute yield/risk tradeoff.

| $c_{opt} - \Delta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 21797 | 6 | 0 | 7 | 0 | 3 | 7 | 6 | 0 | 3 | 7 |
| 21754 | | | | | | 6 | 7 | | | |
| 21705 | | | | 1 | | | | | 0 | 6 |
| 21683 | | | | | | | | | 2 | |
| 21678 | 7 | | 6 | | | | | | | |
| 21673 | | | | | 5 | | | | | |
| 21670 | | | | | 4 | | | | | |
| 21663 | | | | 2 | 0 | | | | | |
| 21647 | | | | | | | | 2 | | |
| 21642 | | | | | | | 5 | | 1 | |
| 21630 | | | | | 2 | | | | | |
| 21624 | | | | | | | | 1,3 | | |
| 21604 | | | | | 1 | | | | | |
| 21599 | | | | | | 5 | | | | |
| 21572 | | | 5 | | | | | | | |
| 21567 | | | | | | | | | 4 | |
| 21562 | | | | | | | | | | 5 |
| 21532 | 5 | | | | | | | | | |
| 21529 | | | | | | | 4 | | | |
| 21484 | | | | | | | | | 5 | 4 |
| 21467 | | | | | | 4 | | | | |
| 21456 | | | | | | | | 4 | | |
| 21412 | | | | | 6 | | | | | |
| 21404 | | | | | | | 3 | | | |
| 21370 | | 1 | | | | | | | | |
| 21351 | | | | | | | | 5 | | |
| 21330 | | | | | | | | | 6 | |
| 21312 | | | 4 | | | | | | | |
| 21232 | | | | | | 3 | | | | |
| 21215 | | | | | | | | | | 3 |
| 21134 | | | | | 7 | | | | | |
| 21133 | | 2 | | | | | | | | |

Hadzic and JNH (2006).

# Decision diagrams

- What if there are **many continuous variables**?
  - Discretize them!
  - Use limited-width **relaxed** decision diagram to obtain optimization bounds.
  - Branch in relaxed decision diagram.
- So far, this method has been applied to IP:
  - Competitive with state-of-the-art IP solvers, or better.
  - Construction of relaxed decision diagram dynamically creates finer granularity for more promising discrete values.

Bergman, Cire, van Hoeve, JNH (2013)

# McCormick factorization

- Can be managed with **global constraints** + **semantic typing**.

Cire, JNH, Yunes (2013).

# Want to know more about CP and optimization?

- See this websites for links to tutorials (slides & videos):

  http://web.tepper.cmu.edu/jnh/slides.html

- See also:

  http://moya.bus.miami.edu/~tallys/integrated.php (CP + optimization)

  http://www.andrew.cmu.edu/user/vanhoeve/mdd/ (decision diagrams)