Postoptimality Analysis with Multivalued Decision Diagrams

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Motivation

• Perform **postoptimality analysis** for 0-1 programming.

• **Problem**: It is hard to reason about the entire solution space.

• **Solution**: Represent the set of near-optimal solutions as a Binary Decision Diagram.
  – This was done in previous work (H&H 2006).
Motivation

• Today’s focus: **Scalability**
  – How large do BDDs grow with problem size?
  – How can we minimize the growth?

• We introduce **sound** BDDs.
  – Much **smaller** than the full BDD.
  – Provide **exact** postoptimality analysis for near-optimal solutions.
Types of Analysis

• Characterization of optimal or near-optimal solutions.
  – How much freedom is there to alter solution without much sacrifice in solution quality?

• Sensitivity analysis.
  – Which problem data significantly affect the solution?

• Online what-if queries.
  – What if I fix certain variables?
Basic Idea

• Use **reduced ordered binary decision diagrams (BDDs)** as a compact representation of the set of feasible or near-optimal solutions.
  – We can extract information from BDDs in real time.
  – Although exponentially large in the worst case, BDDs can be compact for important constraints.
Binary Decision Diagrams

- A reduced ordered BDD for a constraint set is a **compact representation of the branching tree** for a given branching order.
  - If both branches from a node lead to identical subtrees, remove the node.
  - If two subtrees are identical, superimpose them.
Branching tree for 0-1 inequality

\[ 2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7 \]

\[
x_0 = 1 \quad \quad \quad \quad \quad \quad \quad x_0 = 0
\]

\[
x_0 \quad x_0
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x_1 \quad x_1
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x_2 \quad x_2
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x_3 \quad x_3
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\[
1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0
\]
Branching tree for 0-1 inequality

$$2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7$$

$x_0 = 1$
$x_0 = 0$

If an edge skips a variable, both assignments are allowed ($x_3 = 0$ and $x_3 = 1$)
In practice, BDDs are generated **bottom-up**.

First construct a BDD for every constraint and then conjoin the BDDs.

as generated by software (CLab, BuDDy)
• The BDD for a knapsack constraint can be surprisingly small…

The 0-1 inequality

\[
300x_0 + 300x_1 + 285x_2 + 285x_3 + 265x_4 + 265x_5 + 230x_6 + 230x_7 + 190x_8 + 200x_9 +
400x_{10} + 200x_{11} + 400x_{12} + 200x_{13} + 400x_{14} + 200x_{15} + 400x_{16} + 200x_{17} + 400x_{18} \geq 2701
\]

has 117,520 minimal feasible solutions

Or equivalently,

\[
300x_0 + 300x_1 + 285x_2 + 285x_3 + 265x_4 + 265x_5 + 230x_6 + 230x_7 + 190x_8 + 200x_9 +
400x_{10} + 200x_{11} + 400x_{12} + 200x_{13} + 400x_{14} + 200x_{15} + 400x_{16} + 200x_{17} + 400x_{18} \leq 2700
\]

has 117,520 minimal covers

But its reduced BDD has only 152 nodes…
• However, the BDD for multiple constraints can explode.
Optimization Over BDDs

• We want to solve a 0-1 programming model with an additively separable objective function

\[
\text{min} \sum_{j=1}^{n} c_j(x_j)
\]

\[g_i(x) \geq b_i, \quad i = 1, \ldots, m\]

\[x_j \in \{0,1\}, \quad j = 1, \ldots, n\]

Can be straightforwardly extended to general integer programming
Optimization Over BDDs

• We want to solve a 0-1 programming model with an additively separable objective function

\[
\min \sum_{j=1}^{n} c_{j}(x_{j})
\]

\[
g_{i}(x) \geq b_{i}, \quad i = 1, \ldots, m
\]

\[
x_{j} \in \{0,1\}, \quad j = 1, \ldots, n
\]

Can be straightforwardly extended to general integer programming

• If we represent the constraints \( g_{i}(x) \geq b_{i} \) as a BDD, then we can solve the problem by finding a shortest path in the BDD with appropriate edge lengths…
\[
\min 2x_0 - 3x_1 + 4x_2 + 6x_3 \quad \text{subject to} \quad 2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7
\]

Edge lengths reflect coefficients in the objective function.
$$\min \{2x_0 - 3x_1, 4x_2 + 6x_3\}$$ subject to $2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7$

2.1 + min $\{3x_1\}$

Edge lengths reflect coefficients in the objective function.
\[
\begin{align*}
\min & \ 2x_0 - 3x_1 + 4x_2 + 6x_3 \quad \text{subject to} \quad 2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7 \\
2 \cdot 1 + \min_{x_1 \in \{0,1\}} \{-3x_1\} & \rightarrow -1 \\
4 \cdot 1 + \min_{x_3 \in \{0,1\}} \{6x_3\} & \rightarrow 4 \\
\end{align*}
\]
Edge lengths reflect coefficients in the objective function.
Shortest path has length 1
Optimal solution:
\((x_0, x_1, x_2, x_3) = (0, 1, 1, 0)\)

We conduct postoptimality analysis by analyzing shortest and near-shortest paths

\[
\begin{align*}
\min & \quad 2x_0 - 3x_1 + 4x_2 + 6x_3 \\
\text{subject to} & \quad 2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7
\end{align*}
\]
Cost-Based Domain Analysis

• Consider again

\[
\min \sum_{j=1}^{n} c_j(x_j)
\]

\[g_i(x) \geq b_i, \quad i = 1,\ldots,m\]

\[x_j \in \{0,1\}, \quad j = 1,\ldots,n\]

• What values can \(x_j\) take without forcing the objective function value above \(c_{\text{opt}} + \Delta\)?
Example: Network Reliability

- Minimize cost subject to a bound on reliability
  - System of 5 bridges:

\[
R = R_1R_2 + (1 - R_2)R_3R_4 + (1 - R_1)R_2R_3R_4 + R_1(1 - R_2)(1 - R_3)R_4R_5 + (1 - R_1)R_2R_3(1 - R_4)R_5
\]
The problem:

$$\min \sum_j c_j x_j$$

Number of links $j$

$$R \geq R_{\text{min}}$$

$$R = R_1 R_2 + (1 - R_2) R_3 R_4 + (1 - R_1) R_2 R_3 R_4$$

$$+ R_1 (1 - R_2) (1 - R_3) R_4 R_5 + (1 - R_1) R_2 R_3 (1 - R_4) R_5$$

$$R_j = 1 - (1 - r_j)^{x_j}, \text{ all } j$$

$$x_j \in \{0, 1, 2, 3\}$$

Set $R_{\text{min}} = 60$ in all examples
Cost-based domain analysis

308 nodes in BDD
1.1 seconds to compile BDD

\( r = (0.9, 0.85, 0.8, 0.9, 0.95) \)
\( c = (25, 35, 40, 10, 60) \)

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<tr>
<th>( c_{opt} + \Delta )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
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Domain analysis with respect to $R$

Same BDD as before

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7 bridges
Cost-based domain analysis

1779 nodes in BDD
14.8 seconds to compile BDD

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Domain analysis with respect to $R$

Same BDD as before

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
R & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
99.9 & 2,3 & 0,1,2,3 & 1,2,3 & 0,1,2,3 & 0,1,2,3 & 2,3 & 2,3 \\
99.8 & 1 & & & & & & 1 \\
99.5 & 0 & & 0 & & & 1 & \\
99.1 & & & & & & 0 & \\
97.2 & & & & & & & 0 \\
\hline
\end{array}
$$
12 bridges
Cost-based domain analysis

69,457 nodes in BDD
2933 seconds to compile BDD

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Domain analysis with respect to $R$

Same BDD as before

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Reducing BDD Growth

- It suffices to use a BDD with the same near-optimal solutions as the original BDD. We assume $\Delta \leq \Delta_{\text{max}}$. 

$B_{\Delta_{\text{max}}}$
Reducing BDD Growth

- It suffices to use a BDD with the same near-optimal solutions as the original BDD. We assume $\Delta \leq \Delta_{\text{max}}$.

$$\text{Sol} = \left\{ \begin{array}{l} \text{feasible} \\ \text{solutions} \end{array} \right\}$$

$$\text{Sol}_{\Delta_{\text{max}}} = \left\{ \begin{array}{l} \text{feasible solutions} \\ \text{with value } \leq c_{\text{opt}} + \Delta_{\text{max}} \end{array} \right\}$$

$$B = \text{original BDD}$$

$$B_{\Delta_{\text{max}}} = \text{BDD that represents } \text{Sol}_{\Delta_{\text{max}}}$$
Reducing BDD Growth

• It suffices to use a BDD with the same near-optimal solutions as the original BDD. We assume $\Delta \leq \Delta_{\text{max}}$.

$$\text{Sol} = \left\{ \text{feasible solutions} \right\}$$

$$\text{Sol}_{\Delta_{\text{max}}} = \left\{ \text{feasible solutions with value} \leq c_{\text{opt}} + \Delta_{\text{max}} \right\}$$

$$B = \text{original BDD}$$

$$B_{\Delta_{\text{max}}} = \text{represents Sol}_{\Delta_{\text{max}}}$$

• Unfortunately, $B_{\Delta_{\text{max}}}$ can be exponentially larger than $B$.
  – Even though it represents a smaller set of solutions.
Reducing BDD Growth

- It suffices to use a BDD with the same near-optimal solutions as the original BDD. We assume $\Delta \leq \Delta_{\text{max}}$.

\[
\text{Sol} = \left\{ \text{feasible solutions} \right\}
\]

\[
\text{Sol}_{\Delta_{\text{max}}} = \left\{ \text{feasible solutions with value } \leq c_{\text{opt}} + \Delta_{\text{max}} \right\}
\]

\[
B = \text{original BDD}
\]

\[
B_{\Delta_{\text{max}}} = \text{represents } \text{Sol}_{\Delta_{\text{max}}}
\]

- We will construct a smaller BDD $B'(\Delta_{\text{max}})$ that is sound: $B'(\Delta_{\text{max}})_{\Delta_{\text{max}}} = B_{\Delta_{\text{max}}}$
  - It has the same near optimal solutions as $B$. 

\[
\text{Sol} \nonumber
\]
Reducing BDD Growth

\[ \text{Value} > c_{opt} + \Delta_{\text{max}} \]

\[ \text{Value} < c_{opt} + \Delta_{\text{max}} \]

\[ \text{Sol} = \text{Solutions represented by } B \]
Reducing BDD Growth

\[ \text{Sol} = \text{Solutions represented by } B \]

\[ \text{Value} > c_{\text{opt}} + \Delta_{\text{max}} \]

\[ \text{Value} < c_{\text{opt}} + \Delta_{\text{max}} \]

\[ \text{Sol}_{\Delta_{\text{max}}} = \text{Solutions represented by } B_{\Delta_{\text{max}}} \]
Reducing BDD Growth

$Sol = \text{Solutions represented by } B$

Solutions represented by $B'(\Delta_{\text{max}})$

Value $> c_{\text{opt}} + \Delta_{\text{max}}$

Value $< c_{\text{opt}} + \Delta_{\text{max}}$
Pruning and Contracting

• We are unaware of a polytime exact method for constructing the **smallest sound** BDD.

• We use two heuristic methods for generating **small sound** BDDs during compilation:
  – Pruning edges
  – Contracting nodes
Pruning

• Delete all edges that belong only to paths longer than $c_{opt} + \Delta_{max}$.
Pruning

- Delete all edges that belong only to paths longer than $c_{opt} + \Delta_{max}$. 

![Diagram showing the pruning process]
Pruning

• Delete all edges that belong only to paths longer than $c_{opt} + \Delta_{max}$. 

If another edge now belongs only to long paths
Pruning

• Delete all edges that belong only to paths longer than $c_{opt} + \Delta_{max}$.

Delete it, too.
Pruning

• Delete all edges that belong only to paths longer than $c_{opt} + \Delta_{max}$.

And simplify the BDD.
Pruning

• Delete all edges that belong only to paths longer than $c_{\text{opt}} + \Delta_{\text{max}}$.

And simplify the BDD.
Contracting

• Remove a node if this creates no new paths shorter than $c_{opt} + \Delta_{max}$
Experimental Results

• We solve the 0-1 problem

\[
\min c x \\
A x \geq b \\
x \in \{0,1\}^n
\]

\[b_i = \alpha \sum_j A_{ij}\]

\(A_{ij}\) drawn uniformly from \([0,r]\)
### Experimental Results

- 20 variables, 5 constraints
- $C_{opt} = 101$, $C_{max} = 588$

| $\Delta_{max}$ | $|B|$ | $|B_{\Delta_{max}}|$ | $|B'(\Delta_{max})|$ |
|----------------|-------|---------------------|---------------------|
| 0              | 8,566 | 20                  | 5                   |
| 40             | 742   | 524                 |
| 80             | 4,388 | 3,456               |
| 120            | 11,217| 7,034               |
| 200            | 16,285| 8,563               |
| 240            | 13,557| 8,566               |
$B_{\Delta_{\text{max}}}$

Size of BDD

$|B'(\Delta_{\text{max}})|$

$|\text{Sol}| = 72,896$

$|\text{Sol}| = 449,102$

$|\text{Sol}| = 929,260$

$|\text{Sol}| = 556$

$\Delta_{\text{max}} / (c_{\text{max}} - c_{\text{opt}})$

20-5-50-3 $c_{\text{min}} = 101$, $c_{\text{max}} = 588$

Sol = 72,896

Sol = 449,102

Sol = 929,260

Sol = 556
Experimental Results

- 30 variables, 6 constraints
- $c_{\text{opt}} = 36$, $c_{\text{max}} = 812$

| $\Delta_{\text{max}}$ | $|B|$     | $|B_{\Delta_{\text{max}}}|$ | $|B'(\Delta_{\text{max}})|$ |
|-----------------------|----------|-----------------------------|-----------------------------|
| 0                     | 925,610  | 30                          | 10                          |
| 50                    | 3,428    | 2,006                       |
| 150                   | 226,683  | 262,364                     |
| 200                   | 674,285  | 568,863                     |
| 250                   | 1,295,465| 808,425                     |
| 300                   | 1,755,378| 905,602                     |
Experimental Results

• 40 variables, 8 constraints
• $c_{opt} = 110, \ c_{max} = 1241$

| $\Delta_{max}$ | $|B|$ | $|B_{\Delta_{max}}|$ | $|B'(\Delta_{max})|$ |
|---------------|------|-----------------|-----------------|
| 0             | ?    | 40              | 12              |
| 15            | 1,143| 40              | 402             |
| 35            | 3,003| 1,160           | 7,327           |
| 70            | 11,040| 7,327          | 223,008         |
| 100           | 404,713| 223,008      | 52,123          |
| 140           | ?    |                 |                 |
Experimental Results

• 60 variables, 10 constraints
• $c_{\text{min}} = 67$, $c_{\text{max}} = 3179$

| $\Delta_{\text{max}}$ | $|B|$ | $|B_{\Delta_{\text{max}}}|$ | $|B'(\Delta_{\text{max}})|$ |
|----------------------|------|----------------|----------------|
| 0                    | ?    | 60             | 7              |
| 50                   | 5,519| 1,814          |
| 100                  | 111,401| 78,023       |
Experimental Results

Results when tightness $\alpha$ is gradually reduced
Experimental Results

• MIPLIB instances

• $\Delta_{\text{max}} = 0$ (BDD represents all optimal solutions)

| Instance  | $|B|$ | $|B_{\Delta_{\text{max}}}|$ | $|B'_{\Delta_{\text{max}}}|$ |
|-----------|------|---------------------------|-----------------------------|
| lseu      | ?    | 99                        | 19                          |
| p0033     | 375  | 41                        | 21                          |
| p0201     | 310,420 | 737                   | 84                          |
| stein27   | 25,202 | 6,260                  | 4,882                       |
| stein45   | 5,102,257 | 1,765                | 1,176                       |
Conclusions and Future Work

• Cost-bounded BDDs provide reasonable scalability for BDD-based postoptimality analysis in 0-1 linear programming.

Future work:

• Tests on nonlinear, nonconvex 0-1 problems
  – Nonlinearity, nonconvexity should not be a major factor.
• Extension to general integer problems.
  – Straightforward; a matter of implementation.
• Extension to MILP.
  – ??